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Abstract

This study aimed at investigating two main issues related to counterexample construction: the appropriateness of counterexamples and the types of arguments that are often used when refuting a false conjecture. Twelve pre-service elementary teachers who demonstrated a wide range of reasoning skills participated in this study. The data revealed various phenomena among pre-service teachers' conceptions of refutations. While all participants who demonstrated deductive reasoning were aware of the fact that one counterexample was sufficient to refute a false statement, the majority of the participants who had not yet developed deductive reasoning failed to recognize that one counterexample would be enough and/or they tend to believe that providing more than one counterexample would support the argument further to refute the statement. Furthermore, the participants from the deductive proof scheme attempted to construct a general refutation or to provide a justification for their constructed counterexamples while the participants from different proof schemes only provided specific counterexamples without further explanation or justification. This study also documented various misconceptions that PSTs held regarding to the underlying concepts in which they were being asked to refute—the concepts of area and perimeter or quadrilaterals—and how their misconceptions affected their constructed counterexamples or decisions as to whether the presented statements were true or false.

Introduction

The importance of proofs and refutations has been acknowledged as an essential part of mathematics. Consequently, proofs and refutations have gained an increasing level of attention in recent attempts to reform mathematics teaching (CCSSI, 2010; NCTM, 2000). Common Core State Standards for Mathematics (CCSSM) suggest that students should learn to construct counterexamples in order to refute false conjectures at all levels from kindergarten through grade 12 (CCSSI, 2010). Teachers need to be able to refute students' invalid claims to help students develop an understanding of the mathematical situation (Giannakoulias, Mastorides, Potari, & Zachariades, 2010). Despite the fundamental role that proof and refutation play in mathematical inquiry (Lakatos, 1976) and the growing appreciation of the importance of these concepts in students' mathematical education (Reid, 2002), studies demonstrate that students and teachers have difficulty constructing counterexamples (Balacheff, 1991; Giannakoulias, Mastorides, Potari, & Zachariades, 2010; Potari, Zachariades, & Zaslavsky, 2010).

Although teachers' (both in-service and pre-service) justifications and proof strategies have been explored by many researchers (Martin & Harel, 1989; Simon & Blume, 1996; Stylianides & Stylianides, 2009), the process of refutation has not been extensively investigated with regard to the teachers. The aim of this study is to examine the ways that pre-service elementary teachers (PSTs) who demonstrated a wide range of reasoning skills refute false mathematical conjectures. That is, what does it take for PSTs from different proof schemes—specifically, external, empirical and deductive proof schemes— to "disprove" a statement and what kind of counterexamples convinces them that a conjecture is false?

Particularly, the following two research questions guided this study:

1. How do pre-service elementary teachers refute false mathematical statements?
2. How do PSTs' conceptions of refutations differ based on their proof schemes?

Frameworks

Framework to Interpret PSTs' Proof Schemes

Several researchers have attempted to understand students' approaches to mathematical proof by classifying these approaches along several dimensions – an approach currently proving fruitful in understanding students' difficulties (Balacheff, 1988; Harel & Sowder, 1998; Van Dormolen, 1977). Researchers have hypothesized that the development of students' understanding of mathematical justification is likely to proceed from inductive to deductive or from particular cases toward greater generality (Harel & Sowder, 1998; Simon & Blume, 1996).

Harel and Sowder (1998) characterized proof schemes into three major categories as:(1) external proof schemes (2) empirical proof schemes, and (3) analytic proof schemes. External conviction proof schemes are ones in which students convince themselves and others by referring to external sources such as the authority (Authoritarian proof scheme), the format of the argument (Ritual proof scheme), or symbolic manipulation without attending to the meaning of the manipulation (Non-referential symbolic proof scheme). Empirical proof schemes can be either inductive or perceptual. Inductive proof schemes are those that rely on examples or direct measurements. Several researchers have posed questions such as: What might make one example or empirical justification stronger than another? As a result, they have divided inductive justifications into further subcategories as *Naïve Empiricism* and *Crucial Empiricism* (e.g. Balacheff, 1988; Harel & Sowder, 2007; Quinn, 2009). The same approach was followed in this study. Perceptual proof schemes are based on rudimentary mental images that are not fully supported by deduction. Analytic proof schemes are characterized by the validation of conjectures via the use of logical deduction. The analytical proof schemes category was greatly revised by Harel (2007) and renamed as deductive proof schemes. There are two types of deductive proofs: transformational and modern axiomatic. The student considers generality aspects, applies goal-oriented and anticipated mental operations, and uses logical inferences at these levels. In this study, these classifications were used to analyze PSTs' conceptions of proof.

Table 1. Taxonomies of proof scheme

Categories	Characteristics of Categories Subcategories	 Characteristics of Subcategories
External Proof Scheme	Responses appeal to external authority	
	(1) Authoritarian proof	Depends on an authority
	(2) Ritual proof	Depends on the appearance of the argument
	(3) Non-referential symbolic proof	Depends on some symbolic manipulation
Empirical Proof Scheme	Responses appeal to empirical demonstrations, or rudimentary transformational frame	
	(1) Naïve Empiricism	An assertion is valid from a small number of cases
	(2) Crucial Empiricism	An assertion is valid from strategically chosen cases or examples
	(3) Perceptual Proof	An assertion is valid from inferences based on rudimentary mental images
Deductive Proof Scheme	Responses appeal to rigorous and logical reasoning	
	(1) Transformational proof scheme	Involves goal-oriented operations on objects
	(2) Modern Axiomatic proof scheme	Involves statements that do not require justification

Framework to Interpret PSTs' Conceptions of Counterexamples

Potari, Zachariades, and Zaslavsky (2010) argue that refuting conjectures and justifying invalid claims is a complex process that goes beyond the syntactic derivations of deductive proof. Bills et al. (2006) distinguish three special types of examples—generic example, counterexample and non-example. Generic examples are examples of a concept and the core of a generic proof. An example is labeled as a counter-example if that example contradicts a hypothesis or assertion. Finally, an example is labeled as a non-example if they serve to clarify the boundaries where a procedure may not be applied or fails (Bills et al., 2006). Peled and Zaslavsky (1997), on the other hand, categorize the counterexamples according to their explanatory power into specific, semi-general and general ones. Selden and Selden (1998) also adopted these three categories—specific, semi-general, and general— to classify teachers' generated counterexamples. Similar to the distinction between "proof that proves" and "proofs that explains" (Hanna, 2000), counterexamples can be distinguished as follows: (1) a proof by counterexample that shows only that a proposition is false, (2) a proof by semi general counterexample that provides some ideas about why a proposition is false or how the counterexample contradicts the claim, but does not tell "the whole story" and (3) a proof by a general counterexamples provides insight into why a proposition is false and suggests a way to generate not only one counterexample but an entire counterexample space (Peled & Zaslavsky, 1997). These three levels were used to code PSTs' constructions of counterexamples.

Table 2. Counterexample levels

Categories	Characteristics of Categories
Specific Counterexample	Responses appeal to the use of counterexample(s) that show the falsity of the statement
Semi-general Counterexample	Responses appeal to the use of counterexample(s) that show the falsity of the statement and some ideas for why the statement is false
General Counterexample	Responses appeal to generating whole space of counterexamples

Method

Participants

In order to represent a wide range of proof and reasoning abilities a proof questionnaire consisting of sixteen open-ended questions was administered to one section of Geometry and Measurement course and one section of a Mathematics Methods course at a large Mid-western university in the United States. According to the questionnaire results, six[*] pre-service teachers (2 from each major category—external, empirical, and deductive) from a Geometry and Measurement course and seven pre-service teachers (at least 2 from each major category) from a Mathematics Methods course were selected to participate in the individual interviews.

The Geometry and Measurement course and the Mathematics Methods course were selected for the following two specific reasons:

(1) Mathematical proofs and refutations along with geometrical concepts such as area-perimeter or quadrilaterals, which were underlying concepts of the interview tasks, constituted an important part of the Geometry and Measurement course curriculum. Participants of this course were engaged in constructing as well as evaluating mathematical proofs and refutations while they were exploring geometric concepts during the semester. Thus, it was anticipated to detect some changes in PSTs' conceptions of proof and it was aimed to investigate the effects of these changes on PSTs' conceptions of refutations.
(2) Unlike the Geometry and Measurement course, the Mathematics Methods course relied less heavily on mathematical proofs and refutations as well as geometric concepts—area-perimeter and quadrilaterals. However, all participants who enrolled in the Mathematics Methods course already took the geometry course and passed with a passing grade C+ or above. Although it was not anticipated to detect any changes in PSTs' conceptions of proofs in this course since mathematical proofs did not constitute the

[*] One participant completed only pre-interviews. In this study only the results from the participants who completed both the pre- and post-interviews will be shared.

primary focus of the course, it was anticipated that including participants from this course would be a good contrast to the Geometry and Measurement course and might provide further opportunity to investigate the effects of PSTs' conceptions of proof on their conceptions of refutations.

Data Sources

Two main data sources—Questionnaire and Task-based Interviews—were used to investigate the research questions in this study. In this section these data sources and the purpose for using these sources will be described in details.

Questionnaire

A written questionnaire with sixteen open-ended questions was developed to administer to one section of the Geometry and Measurement and one section of the Mathematics Methods course at the beginning of the semester. Questionnaire served to two main purposes in this study: (1) to make sure to select participants who demonstrated varied levels of reasoning skills in terms of mathematical proofs and refutations and (2) to gain background information with respect to PSTs' conceptions of proof and counterexamples as well as their content knowledge. The questions for the questionnaire were designed to assess PSTs' abilities to prove/ refute, to evaluate the correctness of proof (one incorrect and one correct) / a counterexamples presented as well as to assess PSTs' knowledge of content, in particular in the concepts of quadrilaterals and area-perimeter. The questions were adopted and modified from existing literature such as Knuth, 2002 or Kotelawala, 2009 and from high school geometry textbooks (see Appendix A for sample questionnaire questions).

Task-based Interviews

All selected participants based on their responses to the questionnaire questions participated in an hour-long semi-structured task-based interview at the beginning and again near the end of the semester. Participants were interviewed individually by the author at the beginning and again near the end of the semester in order to detect any possible changes in their conceptions of proof and how it might influence their conceptions of counterexamples. During the individual interviews, the participants were provided several tasks that varied in their validity and they were asked to produce a justification in cases where they believed the statements to be true and a refutation where they believed the statements to be false. Three of the tasks that were used included true mathematical statements where the participants were expected to provide a justification. Two of the tasks[†] used included false mathematical conjectures, where the participants were expected to refute the statements. Detailed descriptions of these two tasks will be provided below.

Tasks to Investigate Conceptions of Counterexamples

One task during the pre-interviews and two tasks during the post-interviews were used to understand the ways used by pre-service teachers to refute false mathematical conjectures. In this part, these tasks will be described in detail.

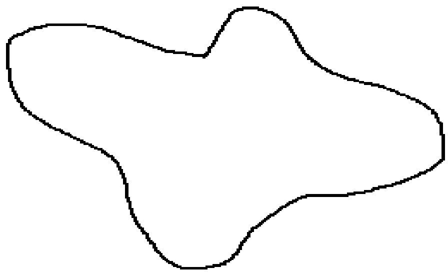
Task A. This task (see figure 1) was designed to analyze, first, pre-service elementary teachers' competence in providing counterexamples, and then, an overview of their ideas as to what constitutes a convincing counterexample. The task was designed to be an unfamiliar case and it was adopted from Simon and Blume (1996). Participants were asked first to calculate the area of the shape in figure 1 and then validate the method presented to find the area of the shape.

Zazkis and Hazzan (1998) defined *reflection questions* as the questions that ask participants to reflect on a solution presented by a third imaginary person and they argue that the benefit of asking reflection questions is to shift the focus from solution to the reason for the solution. After asking participants to find the area of the shape, they were presented a method and asked to validate whether it was a true method. Thus, this task was a

[†] Only one task (Task A), which included false mathematical conjecture, was used during the pre-interviews. Another task (Task B) was added to further investigate how PSTs refute false mathematical conjectures during the post-interviews.

reflection task in a way that participants were asked to reflect on a presented method to calculate the area of a shape.

Find the area of the shape below.



Method:
If you take some kind of string and measure the whole outside of the area and then pull that into a shape like a rectangle, you can easily calculate the area of the figure.
Justify whether or not the above method will work to find the area of the figure.

Figure 1. Task A

Task B. During post-interviews another task was included.

Task B: True or false? In a quadrilateral at least one diagonal cuts the area in half. If true, provide a justification. If false, provide a counterexample.

Figure 2. Task B

This task was adopted from Healy and Hoyles (1998) and was designed to be an unfamiliar case. Zazkis and Hazzan (1998) identified as *familiar with a twist.* The chosen task was familiar in a way that finding or comparing areas of triangles, and constructing diagonals of quadrilaterals or considering diagonal properties, does not present novelty in the assignment. However, the twist was in the idea of congruency and equality.

Data Analysis

Analyzing PSTs' Proof Schemes

Each interview was transcribed, and the interview transcripts and participants' written responses were carefully read, initial impressions summarized, and interesting issues regarding the participants' proof and counterexamples validations highlighted; statement verifications, and proof and counterexample productions as well as evaluations were considered.

The data analysis was started with examining of the transcripts and written work of each participant by using a constant comparative method of coding (Glaser & Strauss, 1967). Interview transcripts were analyzed line by line and internal sets of codes were derived from the data. After the transcripts were analyzed by using a constant comparative method of coding, the transcripts were coded again by using the proof schemes in the conceptual framework as external codes (see Table 1).

The proof schemes described in the framework were used as priori coding scheme and the responses of the participants (their construction and evaluation of mathematical arguments) were categorized based on the source of their conviction. For instance, if the primary source of PSTs conviction laid into providing specific examples, their responses were coded at empirical proof schemes. Stylianides and Stylianides (2009) stated that many of the pre-service elementary teachers who developed empirical arguments were also aware that their arguments were not proof. In this study, PSTs' responses were coded at empirical level not only when they constructed empirical arguments but also they classified empirical arguments when presented as mathematical proofs. In other words, PSTs who were coded at empirical reasoning demonstrated a lack of essential understanding of generality aspect of mathematical proofs. A team of second coders was asked to code a sample of the interview data (20 percent of the interview data). The codings then were compared and discussed until all disagreements were solved.

Analyzing PSTs' Conceptions of Refutations

After participants' proof schemes were determined, their responses to the two tasks (Task A and Task B) that required constructing refutations were analyzed. Firstly, the counterexamples that were constructed by the participants were coded as relevant or irrelevant based on whether the counterexample(s) were relevant or irrelevant to deduce the falsity of the statements. Then, relevant counterexamples were categorized into three categories—specific, semi-general, and general—based on their explanatory power. The participants' responses to the tasks A and B were also compared with others in the same proof schemes in order to investigate similar patterns across the data. A team of second coders was again asked to code a sample of counterexamples that were constructed by the participants. The codings, then, were compared and discussed if there were any disagreements in the codings of the counterexamples. It should be noted here that the procedures of analyzing PSTs' conceptions of proofs and counterexamples as described above were also applied to analyzing their responses to the Questionnaire questions.

Results and Discussion

In this part, PSTs' responses to the two tasks—Task A and Task B—(see Figure 1 and 2) will be reported in details. Even though the main focus of this article is on PSTs' conceptions of counterexamples and their ways of refuting false mathematical conjectures, their proof schemes are important: (1) to illustrate how PSTs from different proof schemes refute false mathematical conjectures; (2) to provide a full picture of participants' reasoning skills to justify mathematical conjectures. Thus, in the following section PSTs' proof schemes will be displayed in a summarized table (see Table 3) along with a sample of their responses to the questionnaire questions in order to explain how PSTs' conceptions of proof were categorized according to the framework.

Participant Profiles

According to the questionnaire and task-based interview results, participants demonstrated a wide range of abilities in terms of their proof schemes. Table 3 below summarizes participants' proof schemes in a cumulative way based on PSTs' responses to the questionnaire questions and the interview tasks. Then, PSTs' responses to sample questionnaire questions will be presented next in order to demonstrate how the participants were selected and how the frameworks to investigate PSTs' conceptions of proof and counterexamples were applied to questionnaire analysis.

Table 3. Pre-service teachers' proof schemes

Categories		Participants in Each Category			
		Pre-Interviews		Post-Interviews	
		Geometry & Measurement	Mathematics Methods	Geometry & Measurement	Mathematics Methods
External Proof Scheme		Elizabeth	Elaine Laura Daisy	Elizabeth	Elaine Laura
Empirical Proof Scheme	Naïve Empiricism	Chloe Sara	Casey Laura Daisy	Sara	Casey Laura Daisy
	Crucial Empiricism	Kelly			
Deductive Proof Scheme		Scott	Jack Miranda Mary	Chloe Kelly Scott	Jack Miranda Mary

Participants' Responses to the Questionnaire Questions

In this part PSTs' responses to a sample of the questionnaire questions will be demonstrated in order to provide the reader with a clear understanding of how the framework is applied. Participants' responses to the questionnaire questions were categorized into three main categories—external, empirical or deductive—based on the source of their conviction. If PSTs' primary source of conviction resulted from external reasons such as the format of the argument as it was in the case of Elizabeth, PSTs' responses were coded as external reasoning. In the example below (see figure 3), Elizabeth argued that the presented proof (see question 3 in the questionnaire in appendix A), which was an incorrect argument, could be classified as a correct mathematical proof. In addition to not recognizing the logical flaw in the presented argument, Elizabeth classified the argument as a mathematical proof solely based on its presented format. It was evident in her response in figure 3 that a mathematical proof should consist of steps and use formal language such as using triangle congruency. Yopp (2015) argues that PSTs' claims are influenced by their use of language. In this case, it was evident that seeing mathematical language was used was an important criterion for Elizabeth when evaluating mathematical arguments.

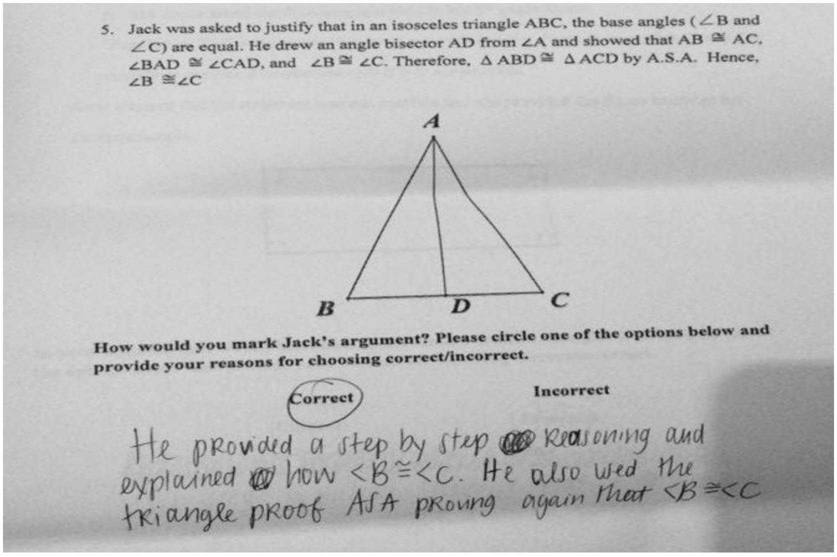


Figure 3. Elizabeth's response to a questionnaire question

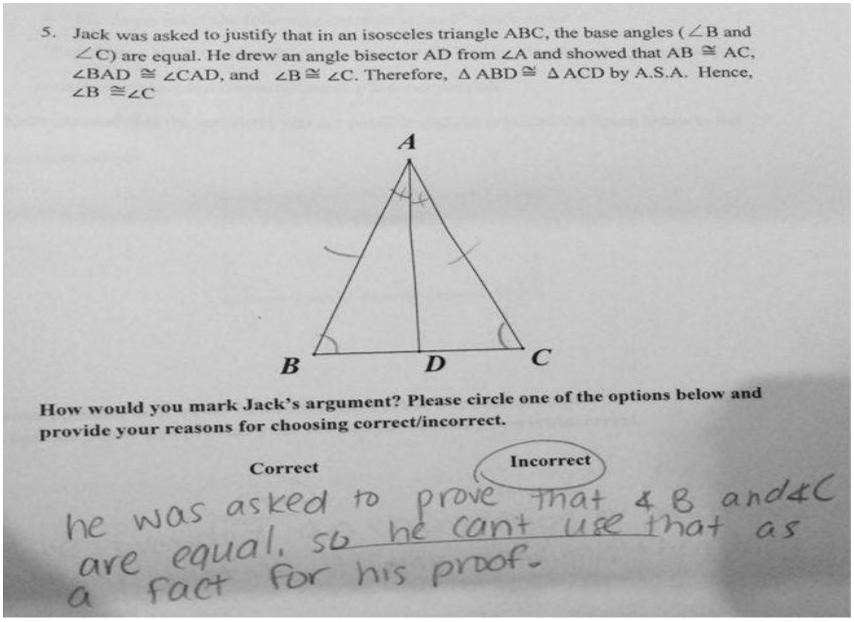
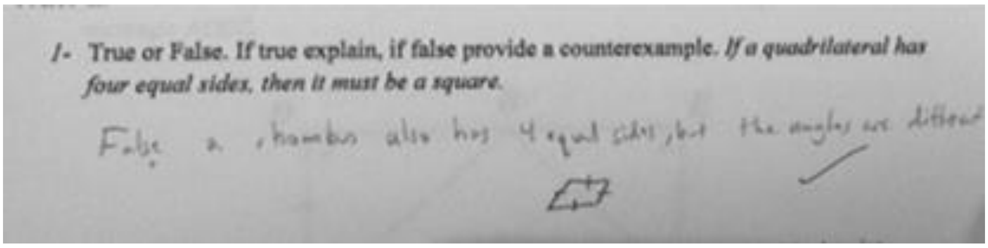


Figure 4. Jack's response to a questionnaire question

Unlike Elizabeth, Jack was able to detect the flaw in the logic of the presented argument (see the questionnaire question 3 in appendix A) and classify the argument as an incorrect argument. Several researchers have argued that the reading of proof has received relatively little attention despite the fact that there is a growth in educational research on mathematical proof (Alcock & Weber, 2005; Selden & Selden, 2003). The way a person evaluates a mathematical argument is necessarily dependent on how he or she removes personal doubts about the truth of mathematical statements (Alcock & Weber, 2005). It was evident when Elizabeth read the presented argument; she focused on whether the argument had the appearance of mathematical proofs that she had seen in the past, such as the use of steps, the appearance of mathematical symbols or formal language in geometry (see figure 3). However, when Jack read the same argument it was evident that he focused on whether the content of the argument made sense. Thus, his response in figure 4 was coded as deductive proof scheme in this question.

In addition to consructing and evalutating mathematical proofs, PSTs were also asked to refute false mathematical statements as well as to evalute presented counterexamples in the questionnaire (i.e. questionnaire question 1 in appendix A). In the following response in figure 5, Scott was able to evalate the correctness of the presented mathematical statement and refute the statement by presenting a counterexample. His counterexample was coded as a semi-general counterexample since he justified why his counterexample contradicted the statement as opposed to solely providing his counterexample.


1- True or False. If true explain, if false provide a counterexample. *If a quadrilateral has four equal sides, then it must be a square.*

False a rhombus also has 4 equal sides, but the angles are different

Figure 5. Scott's response to a questionnaire question

PSTs' conceptions of proof, which were displayed in Table 3, were determined based on their responses to the questionnaire questions as well as the individual interview tasks. Since the primary purpose of the study is to investigate how PSTs refute false mathematical statements; only PSTs' responses to the two interview tasks (see Task A and Task B) will be presented in details next.

Participants' Conceptions of Refutations

The main purpose of this study is to investigate what it takes pre-service elementary teachers to refute false mathematical statements which also includes what kind of counterexamples PSTs find convincing. In this section, how PST chose to refute false mathematical statements and their constructed counterexample(s) in order to refute these false mathematical statements will be described. Table 4 below summarizes PSTs' counterexample levels.

Table 4. Pre-service teachers' counterexample levels

Categories	Participants in Each Category			
	Pre-Interviews		Post-Interviews	
	Geometry & Measurement	Mathematics Methods	Geometry & Measurement	Mathematics Methods
Specific Counterexample	Elizabeth Chloe Kelly Sara	Elaine Laura Daisy Casey	Elizabeth Sara Chloe Kelly	Elaine Laura Daisy Casey
Semi-general Counterexample	Scott	Miranda Jack Mary		Miranda Mary
General Counterexample	-	-	Scott	Jack

Pre-Interview Results

Geometry and measurement course. Participants from the Geometry and Measurement course—Sara, Scott, Kelly, Elizabeth, and Chloe (all pseudonyms)—demonstrated a wide range of reasoning abilities regarding to mathematical proofs in the individual interviews, as can be seen in Table 3 above. Scott was the only participant from the Geometry and Measurement course who demonstrated deductive reasoning during the pre-interviews. Elizabeth demonstrated external proof scheme and the rest of the participants demonstrated empirical proof scheme (Sara and Chloe demonstrated naïve empiricism while Kelly demonstrated crucial empiricism).

In this part, how PST from the Geometry and Measurement course refuted false mathematical conjectures with respect to their proof schemes will be discussed. When task A was presented, all participants except Scott believed that the method would work without generating examples. Besides their proof schemes, another reason for why PSTs did not choose to generate example to test the method might be PSTs' limited content knowledge about area and perimeter. There is evidence in the literature that pre-service teachers confuse perimeter and area and they believe that there is a relationship between perimeter and area (Baturo & Nason, 1996; Fuller, 1997; Heaton, 1992).

> Elizabeth: I do not know. I think I would not be able to do it (referring to finding the area of the shape). I do not even know how I could measure it (referring to the shape).
> Interviewer: If you were able to use a ruler, protractor or any other measurement tool, would that help you to find the area?
> Elizabeth: Maybe. Because of the lines, I do not know how I could measure the lines. (Pause). I would try to measure the lines (referring to measuring the outside of the shape), and then find the area, not the perfect area, though.
> Interviewer: What would you do next after you measure the lines?
> Elizabeth: And then times them (the measurement of the outside of the shape). No, wait! I will add them. It probably would not be exact, but it would be an estimate if I add all the lengths of the lines up.
> Interviewer: What would that give you?
> Elizabeth: The area of this shape. Well, not the perfect area.

As can be seen in the excerpt, Elizabeth confused area and perimeter. She suggested measuring the lines that surrounded the shape and add them up to find the area. Later, when the method for task A was presented, she believed that it would work. Unlike Elizabeth, Sara was aware of the difference between area and perimeter. However, Sara believed that there was a relationship between perimeter and area. It is well evidenced in the existing literature that pre-service teachers believe that there is a constant relationship between area and perimeter (Baturo & Nason, 1996; Simon & Blume, 1994).

> Interviewer: How did you decide that this method would work?
> Sara: Um, well, if you took the string, you can measure the outside of the shape. If you put it into a rectangle, you could measure the sides easily and then you just know that the area of a rectangle is length times width. So, you can just easily find the area from there.
> Interviewer: When you measure the outside of this shape with the string, what are you measuring?
> Sara: You are measuring the perimeter.
> Interviewer: How do you know that if the perimeter is the same, then the area would also be the same?
> Sara: Because if, well, because like the space inside of this (the shape) would be equal to the space inside of the rectangle since they have the same outside. Perimeter and area are related! The bigger the perimeter, the bigger the area!

Stavy and Tirosh (2000) describes an intuitive rule—More A- More B—, which evolves from experiences in everyday life. In this intuitive rule, a perceptual quantity (A) can serve as a criterion for comparing another quantity (B). They argue that this rule is common core to many misconceptions in science as well as in mathematics. In this case, Sara was arguing that if the perceptual quantity –perimeter– is bigger, then another quantity –area–will also be bigger, which indeed prevented her from evaluating the presented method.

All participants except Scott believed that the method would be sufficient to calculate the area of the shape in task A without generating any examples to check the validity of the method. Scott, on the other hand, questioned whether the method would work. In order to do that, he chose to generate examples (see figure 6).

> Interviewer: Do you believe that this method might work?
> Scott: So if you take a string and do the outside, that's a perimeter. And (pause) I am thinking if area and perimeter are related, whether you could calculate the area of a rectangle by taking a string and forming that into a rectangle. I wonder if you formed it into a triangle, whether it would give you the same area. I guess I would just try to see if it works for other figures. So, let's try. If the whole thing was 12 cm around, then I could draw a rectangle with 4 sides of 3. I guess it does not need to be exact.

This would be a square, but a square is a rectangle (drawing a square of side 3 in figure 6). Then, this would have the same length. It would be 4,4,4 (drawing an equilateral triangle with the side of 4 in figure 6).

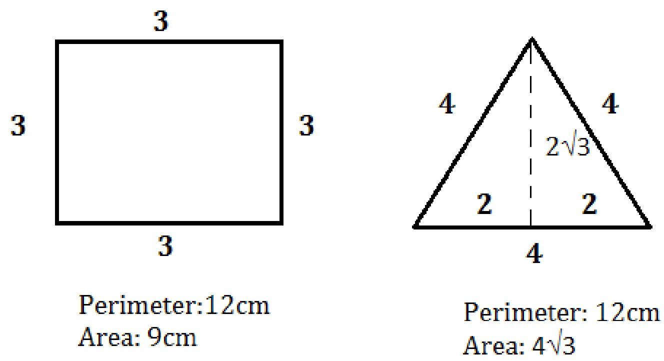


Figure 6. Scot's counterexample

Scott: (Calculating the areas of the shapes he drew in figure 6). These are all the same (referring to the sides of the triangle in figure 6). This is 90 degrees, these are 60,60 (dropping a perpendicular to one of the sides of the triangle). So, in order to find the area of a rectangle, length times width. It (the area of the square) is 9 cm^2. I am trying to find the area of this triangle. It is ½ base times height, so we have to find the height. This makes 2 and 2 (he dropped a perpendicular from the top vertex to the opposite side which also bisected the opposite side). So, $2^2 + h^2 = 4^2$ (applying the Pythagorean theorem to find the height of the triangle he constructed into the equilateral triangle). The height of the triangle is $2\sqrt{3}$ and it makes the area ½ .4. $2\sqrt{3}$ (calculating the area of the triangle). The area of the triangle is 4 √ 3.The area of the square is 9 and the area of the triangle is 4 √ 3.No, it is not working! Both the square and the triangle have the same perimeter, but yet different areas. So, area and perimeter are not related.

Dahlberg and Housman (1997) stated the benefits of example generation by evidencing that the students who generated examples were the ones best able to identify the correctness of conjectures and provide explanations. They also found that the students who primarily reformulated concepts without generating examples were more easily convinced of the validity of a false conjecture. The present study aligned with the results of Dalhberg and Housman's study by concluding that participants at the deductive level attempted to generate examples to identify the correctness of the statements while others were easily convinced of the validity of a false conjecture. Additionally, Scott provided a semi-general counterexample by explaining why these examples contradict to the claim. Even though he did not attempt to cover the whole space of counterexamples or explain the condition when the area and the perimeter of two shapes do not correlate, he attempted to explain his constructed counterexamples.

Watson and Mason (2005) coin the term *example space* to refer to the collection of examples that fulfill various kinds of functions. Bills et al. (2006) argue that the appropriateness of counterexamples for a person should be understood in terms of his/her possible example spaces, thus, in return may provide insight into that person's possible example spaces. It was evident in the study that generating counterexamples encompassed examples that were accessible to the participants in response to a particular situation. For instance, when Scott attempted to generate a rectangle with the perimeter of 12 cm and drew a square of side of 3 cm, it was evident that he was reasoning inclusively and seeing square as a rectangle.

Another difference between participants who demonstrated deductive reasoning and those who did not was deciding whether one counterexample would be sufficient to disprove the statement. The majority of the participants—Elizabeth, Kelly, and Chloe—who did not demonstrate deductive reasoning believed that one counterexample would not be enough to refute or argued that providing more than one counterexample would make their argument stronger. However, it should be noted here that even though the participants argued that providing more than one counterexample would be necessary or make the argument stronger, they all were convinced that the method would not always work after seeing one counterexample.

As can be seen in the following quote, Elizabeth argued that providing more than one counterexample could make an argument more convincing.

Elizabeth: I mean the more the merrier. I would maybe give more than one example, but I think this example shows that does not work.

Similarly, Kelly argued one counterexample would not be enough to refute a statement and providing more counterexamples could be more convincing to refute a statement.

Kelly: I would probably try to show a couple shapes, um, not just one rectangle. I would try to show rectangles in different sizes.
Interviewer: How many rectangles would you show?
Kelly: I mean I would show a hundred of them if I wanted to, but it would be a waste of time (smiles), take a lot of time (Pause). I would probably show six examples, because a hundred is hard to do and showing just two examples is too little (smiles).

Mathematics methods course. Similar patterns were observed among participants from the mathematics methods course. In the mathematics methods course, three participants—Mary, Jack and Miranda—demonstrated deductive reasoning, Casey demonstrated empirical reasoning and Elaine demonstrated external proof scheme during the pre-task based interviews. Two participants—Laura and Daisy—demonstrated the characteristics of both empirical and external proof scheme, so they were coded at two levels.

When task A was presented, all participants except the three participants—Mary, Jack, and Miranda, who demonstrated deductive reasoning—believed that the method would hold true without attempting to generate examples to check the validity of the method. One of the reasons was that participants believed that there was a relationship between area and perimeter.

All participants who had not yet developed deductive reasoning, except Laura, argued that one counterexample would not be enough to refute the statement. Those participants also claimed that providing more than one example would strengthen the argument to refute the statement.

When presented the method for task A, Elaine thought that the method would work to calculate the area. Then, she attempted to generate examples not to refute the statement, but to justify that the method would hold true (see figure 7).

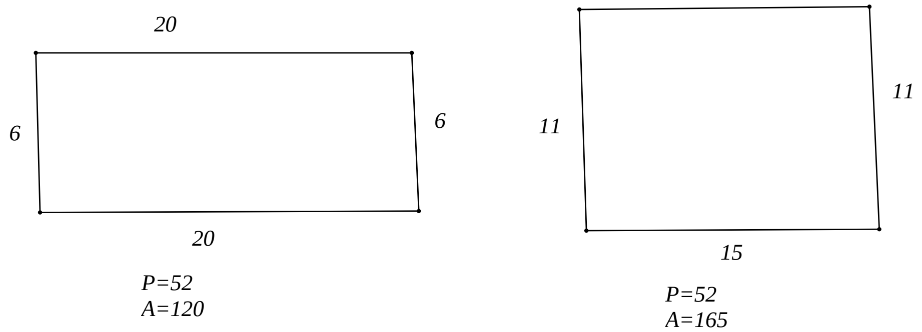


Figure 7. Elaine's counterexample for task A

Interviewer: Do you think that this method would work?
Elaine: Uh huh, yes.
Interviewer: Ok, how would you convince someone that this method would work to find the area of a shape like this?
Elaine: Um, I would probably do two rectangles. Um, I guess would do (drawing the rectangles in figure 7). (Pause) No, wait, it does not work. It is false!

As can be seen in the excerpt, Elaine drew 20 by 6 and 15 by 11 rectangles in order to justify the method, which resulted in recognition of the invalidity of the method. After realizing that the method would not always hold true, she refuted the method. However, she also said that she would provide several counterexamples to refute the method.

Interviewer: So, how would you convince someone that this method would not work?
Elaine: I would draw different shapes, like a triangle, circle, um, a square with the same perimeter and then try to find their areas.

> Interviewer: If this was your exam question, analyzing whether this method was true or false, and you said that it was false and provided these examples (referring to 20 by 6 and 15 by 11 rectangles), do you think that you would get full points for the question?
> Elaine: Um, I would definitely try to put more examples if this was in the exam. I think that I would just support the answer more!

Elaine concluded that the method would not work after being able to construct a counterexample; however, she believed that she would provide more counterexamples if it was an exam question in order to make the argument stronger. Harel and Sowder (1998; 2007) define a proof as an argument that a person uses to convince himself or herself and others of the truth or falseness of a mathematical statement; thus, it consists of what constitutes "*ascertaining*" and "*persuading*" for a person or community. They use the terms ascertaining and persuading to describe complementary processes of proving as convincing oneself and convincing others. It was evident in this study that even though finding one counterexample is convincing for themselves to refute the method, the participants attempted to provide more counterexamples when it comes to convince others.

Casey also argued that more counterexamples could make an argument more convincing.

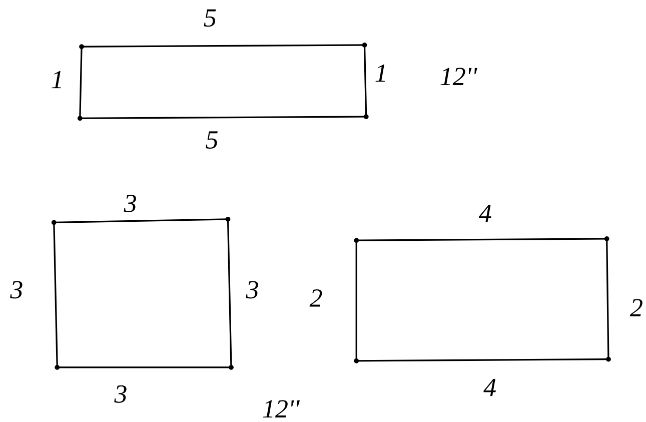


Figure 8. Counterexamples constructed by Casey

> Casey: Um, I mean I would think two would be enough to show that it would not work every way you do it, but the more (examples) you have would be more convincing. The more you have, the more convincing it will be!

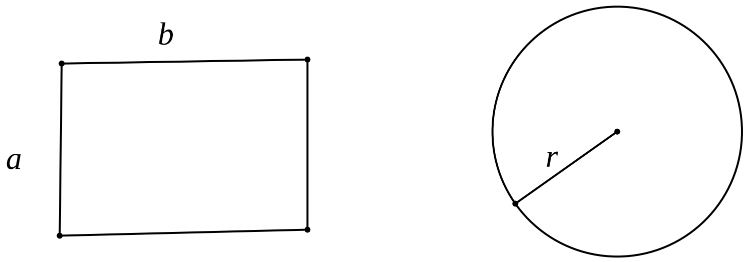
Similar to Scott from the Geometry and Measurement class who demonstrated deductive reasoning, Miranda, Jack, and Mary from the mathematics methods course also questioned whether the method would work and checked the validity of the method by generating examples.

> Miranda: (Reading the method) I do not know if this would work (laughs).
> Because it is assuming that the perimeter of the object is what corresponds with the area, but I do not think that is true.
> Interviewer: How did you decide that it is not true?
> Miranda: Um, well, I drew another shape in my head. It was like, because it has a really long perimeter, but would not have a lot of area, cause it is really skinny, so if you like measured out the lines it would not necessarily, like, correspond with the area.

Mary, Jack, and Miranda were also able to construct an appropriate counterexample when presented with an incorrect mathematical statement. Furthermore, they all were able to state that providing one counterexample would be enough to refute the statement.

When Jack was presented with task A, he questioned whether the method would work. He constructed a 4 by 2 rectangle and a circle with perimeter of 12 (see figure 9). He realized that the areas would differ and concluded the method would not work. Even though Jack argued that one counterexample would be enough to refute a statement, he also argued that providing a generic counterexample would be more convincing than providing a specific counterexample.

He suggested using the following argument to refute the use of the method presented for task A (see figure 9):



$$2\pi r=2(a+b)$$
$$\pi r=(a+b)$$
$$r=\frac{(a+b)}{\pi}$$

$$a.b=\pi r^2$$
$$a.b=\pi \frac{(a+b)^2}{\pi^2}$$
$$a.b=\frac{(a+b)^2}{\pi}$$

Figure 9. Jack's counterexample for task A

> Interviewer: How many counterexamples you would you provide?
> Jack: Um, well, I think this would almost be a good enough counterexample
> Interviewer: A circle and a rectangle
> Jack: Uh huh, so these are two possible shapes that you could have made. And so, honestly, the best counterexample would be a general case that would get messy, but, if I want to give you, I have to give you a, I can just, you know a, b, these are the same so the perimeter of the rectangle is 2a+2b. That's perimeter, but the area is a·b. And then, I drew a circle with some radius that I have not found yet and I am wanting to prove that they have to, the perimeters have to be the same, so 2a+2b equals 2 .**π**.r, so divide by 2. So, a+b divided by **π** would be my radius. And then, to find the area, I have to use **π**·r^2 and that would give me $(a+b)^2$ over **π** after, I just solved that one from square the top and the bottom, **π** is canceled, so this over **π**. I don't, I do not believe that would give you a·b.

Jack was the only participant who was concerned about the generality of his counterexample during the pre-interviews. Even though the other participants from deductive proof scheme attempted to explain how they generated their counterexamples and why their examples refute the statement, they all provided semi-general counterexamples with specific measurements and explained why those examples contradicted the claim during the pre-interviews. Jack, however, attempted to construct a counterexample that can be generalizable to other cases and contradicts the claim by using variables (a,b)instead of specific numbers (see figure 9).

Post Interview Results

Geometry and measurement course. Chloe's and Kelly's conceptions of counterexamples showed differences in post-task-based interviews. Unlike pre-task-based interviews, both Kelly and Chloe were aware that only one counterexample would be needed to refute the method presented for task A. In the following excerpt, Kelly needed to construct more than one counterexample in order to convince herself that the method would not always hold true. However, she was able to state that only one counterexample would be enough to disprove the statement.

> Kelly: I would say, I mean I just think that since I know that the, that a rectangle is, I mean, let's say that this is the rectangle and it's three by two since I'm not changing the length of what this was I'm just putting it into a shape that I can calculate (if) it'll be true (drawing the rectangles in figure 10).

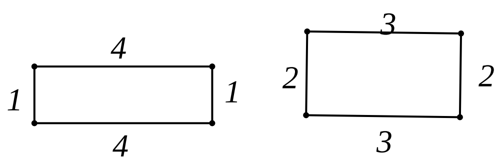


Figure 10. Counterexample generated by Kelly

> Interviewer: Do you think that the area of those rectangles (4 by 1 and 3 by 2 rectangles) will be the same?
> Kelly: The area, I don’t know, yeah it should be, three times two is six, four times one is four though why is that happening, no! I don't know that would be six. (Pause). The side lengths equal ten; it will work but I don't know why it's not doing that for, I'm confusing myself so as long as I measure this to be ten, no that's perimeter, the length is ten but it [area] won't be the same if you're using the same string.

After realizing that the method would not always work, Kelly wanted to check further examples to convince herself that the method was indeed not valid. Zazkis and Chernoff (2008) introduced the notions of pivotal example and bridging example and highlighted their roles in creating and resolving cognitive conflict. It was evident in this excerpt that the rectangles in figure 10 created a cognitive conflict for Kelly; however, they were not enough to resolve the conflict since Kelly attempted to construct more examples, as can be seen in figure 11, instead of refuting the method right away.

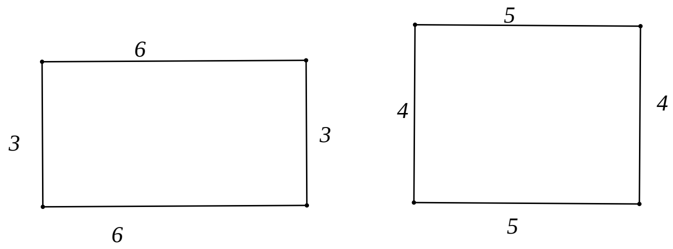


Figure 11. The bridging examples that Kelly constructed

> Kelly: It should work, I thought it would work, hold on. Ok, I'm going to try five by four is going to be twenty and that so that's eighteen and now that's eighteen, no it won't always work (drawing rectangles in figure 11). This is so bad! This is like rocking my world right now!
> Interviewer: How would you use disprove this method works if this was your exam question?
> Kelly: I would just draw this (referring to the rectangles in figure 11)!

Kelly realized that the method would not work and she explained that providing one counterexample would suffice to refute the method. In the pre interviews, however, she argued that more counterexample would be needed. Thus her conception of refutation was changed between the pre and post interviews along with her conceptions of proofs. It should be reminded that she demonstrated empirical reasoning during the pre-interviews; however, she demonstrated deductive reasoning during the post interviews. Her counterexample that she described she would provide if it were the exam question was coded as specific counterexample.

When Scott, on the other hand, who demonstrated deductive reasoning during the pre as well as the post interviews, was presented with task A, he was able construct a 2 by 2 square and a 3 by 1 rectangle with perimeter of 8 to refute the method (see figure 12). Even though Scott argued that one counterexample would be enough to refute the method, he searched for further explanation for why the method would not always work as can be seen in the excerpt below.

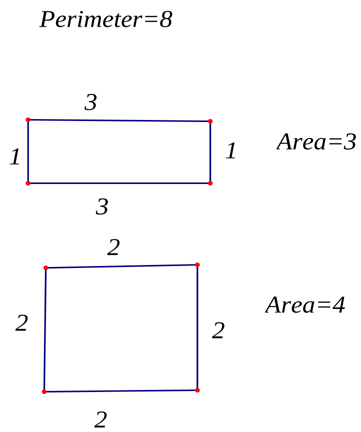


Figure 12. Scott’s refutation for task A

Scott: Um, this method will not always work. You can make two rectangles with the same perimeter (drawing rectangles in figure 12). I did 2 by 2 rectangle, well it is actually a square but squares are rectangles, and 3 by 1 rectangle. They have the same perimeter, 8, but different area. The square has bigger area than the rectangle. It is, um, I think the area will increase if your shape looks more like a square. Like, if the perimeter is 12 then you can make a 3 by 3, 2 by 4, and 1 by 5 rectangles (drawing the rectangles in figure 13). Well actually you can make many rectangles with the same perimeter and when the shape is least like the square, the area will be lowest. The area of the square is 9 and the area of the 5 by 1 rectangle is only 5.

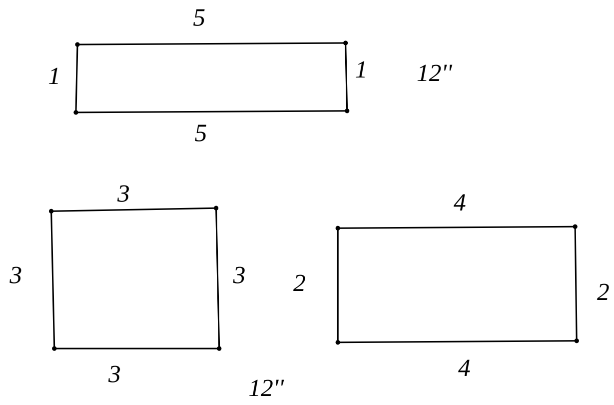


Figure 13. Scott's argument for task A

Interviewer: Ok, how would you justify that the method would not always work if this were your exam question?
Scott: I would use these two rectangles (rectangles in figure 12). Because if you disprove it once then it disproves it; giving one example is enough to disprove a statement!

Scott knew that one counterexample was enough to show the falsity of a statement. However, he went further by showing a way to generate whole sets of counterexamples by saying "Well actually, you can make many rectangles with the same perimeter and when the shape is least like the square, the area will be lowest…"as opposed to finding just one relevant counterexample. Ma (1999) investigated teachers' from two countries (the U.S. and China) reactions to the claim of " if the perimeter increases, the area increases" and documented that very few teachers (none from the U.S.) considered to further investigate the cases when this claim holds true or does not hold true while relatively more teachers tented to provide a counterexample to refute the claim without further investigation of under what condition(s) the claim holds true. It was evident in the excerpt above that Scott was able to construct relevant counterexample that suffices to refute the method. However, he aimed to further investigate the cases when the claim does not hold true. He was aware that he could draw many rectangles with the same perimeter and the areas of these rectangles will fall into a range since the gap between the length and the width gets smaller the area gets bigger. Thus, his argument for task A was coded as general counterexample.

During the post-interviews, the participants were also shown task B in addition to task A to further analyze their conceptions of counterexamples. Including task B also provided insight into participants' example spaces regarding quadrilaterals, which will be discussed next. When task B was presented, Sara was able to recognize that it would not hold true for all quadrilaterals and construct an appropriate counterexample to refute the statement.

Sara: I want to say sometimes because I drew this quadrilateral and that's not half (drew the quadrilateral that is not circled in the figure 14).
Interviewer: Ok, so the diagonal is not cutting it in half. How about the other diagonal? Do you think that it would cut it in half?
Sara: No, but if you drew like a kite and drew the diagonal then it cut that in half, but in this shape it does not.
Interviewer: How would you prove that the task is not true?
Sara: By drawing a quadrilateral (drawing the quadrilateral that is circled in figure 14).
Interviewer: Do you think that this example that you circled enough to prove that the statement is not true?
Sara: Yeah!

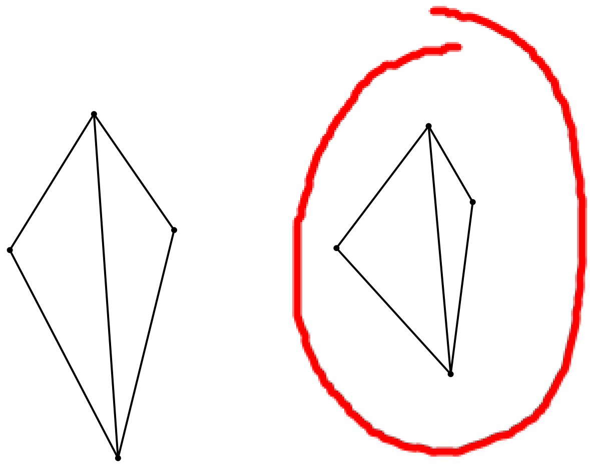


Figure 14. Sara's counterexample for task B

Even though Sara was able to construct a relevant counterexample to show that the statement would not always hold true, she did not attempt to provide a justification for why it could be a relevant counterexample. Instead, she depended on the visual appearances of the areas of the two parts in the quadrilateral not being the same since she selected the one in which this visual difference is more severe (see figure 14). This applied to other participants from different proof schemes. Thus, her counterexample was coded as specific counterexample for this task.

Mathematics methods course. When task B was presented, Elaine was able to recognize that it would not always hold true. Additionally, she was able to construct a sufficient counterexample where the statement would not hold true as can be seen in the following excerpt.

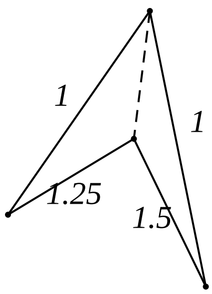


Figure 15. Elaine's counterexample for task B

> Elaine: Ok I think this is sometimes true, because like obviously you can do it with a square, so it would work for it. If you just had like some random quadrilateral then I don't think it would work.
> Interviewer: Ok, how did you decide that it would not always work?
> Elaine: Well I'm just assuming, because a quadrilateral just has to have four sides, so I mean it could be like any type of, like you could just have random shapes; so I guess that would be, umm, I think it will only work for some quadrilaterals.

When asked to describe how she would disprove the statement, Elaine replied as follows:

> Elaine: Proving that it is not true, showing at least one example where it doesn't work.
> Interviewer: What would be your example if this were your exam question?
> Elaine: I'm trying to think of how to make sort of random, ok so say you have that shape like (drawing the shape in figure 15), I'm not even sure where the diagonal would be so it wouldn't work.
> Interviewer: How do you know this will not work for that shape?
> Elaine: Because even though you're cutting it in half, it's not equal parts.

It should be noted here that Elaine argued during the pre-interviews that providing more than one counterexample would make an argument more convincing, even though she was aware that one counterexample would be enough to refute a false statement. However, during the post- interviews, she did not even mention providing more than one counterexample to make an argument more convincing. Additionally, Elaine was able to provide an appropriate counterexample to refute the statement. Even though she attempted to put numbers to show that the areas would not be the same, she did not produce a full justification for why the areas would indeed not be the same. Thus, her counterexample was coded as specific counterexample.

Watson and Mason (2005) introduce extreme examples as extreme cases that show what happens at the "edge" of classes (p.100). It was evident in the excerpt that concave quadrilaterals take places in Elaine's example space. Furthermore, they might serve as an extreme example since Elaine attempted to use a concave example to show that the statement would not hold true. When task B was presented, Miranda was able to propose a counterexample—a trapezoid—to refute the statement and she was able to explain why a trapezoid would be a sufficient counterexample for task B.

> Miranda: I think this (referring to task B) is not always true, cause it doesn't work, for instance, for a trapezoid. I was thinking that it would work for shapes in which it makes it like symmetrical, but if it were a trapezoid it wouldn't make symmetrical parts when you cut in half.
> Interviewer: Ok, do you think that the shapes should have symmetrical parts for this statement to be true?
> Miranda: Not necessarily, because just because things aren't symmetrical doesn't mean they can't have the same area. If you had a trapezoid and cut it like this, well at least one diagonal, well for this diagonal [AC] it wouldn't be equal because you can just see that one side's bigger than the other but for this side it still wouldn't be equal. Because even though the bases would be the same for each triangle, the heights would be different for the triangles that the diagonal constructed in the trapezoid.

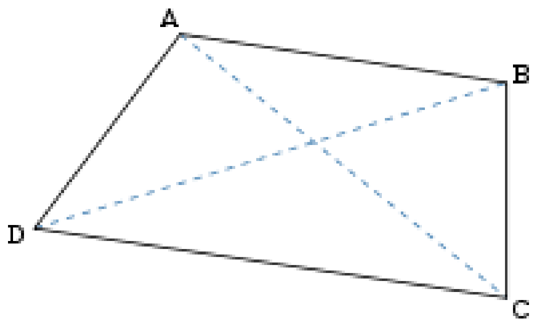


Figure 16. Miranda's counterexample for task B

As can be seen in the excerpt, Miranda attempted to investigate the condition when the statement holds true or when it does not. She proposed the idea of a diagonal (at least one) being a line symmetry (or creating symmetrical parts in the shape) in order for the task to be true. However, she was able to recognize that areas of the shapes that are not symmetrical could still be the same. She revised her condition and explained that if the triangles formed by a diagonal had different heights even if the bases were the same, the areas would be different. Even though she was not able to describe the condition when or in which cases this statement does not hold true, she was able to construct a counterexample as in figure 16 and provide a justification to explain why the statement did not hold true for that specific counterexample. Thus her counterexample was coded as a semi-general counterexample.

Conclusion

This study aimed to investigate what patterns could be observed among pre-service teachers who demonstrated different reasoning skills when they attempted to refute false mathematical conjectures. Although this study aimed to investigate how the types of arguments that were often used when refuting false statements by PSTs who demonstrated a wide range of reasoning abilities differed, the findings of this study suggested that the ways PSTs' refute false mathematical statements and their conceptions of counterexamples showed similarities in two proof schemes--empirical and external proof schemes. However, PSTs' conceptions of counterexamples differed significantly in deductive proof scheme. How PSTs, who demonstrated deductive reasoning and did not demonstrate deductive reasoning (demonstrated empirical or external reasoning instead), chose to refute false mathematical statements and what patterns were observed among different proof schemes will be shared next.

A general proof, covering all relevant cases, is necessary to validate the statement and a single counter example is sufficient to refute the statement. This study showed that the PSTs who were aware of the necessity to cover all possible cases in order to prove true mathematical statements were also aware that providing one counterexample was sufficient to refute a false statement. However, this finding is not bidirectional. That is, if PSTs are not at deductive proof scheme, it is not necessarily mean that they cannot recognize that one counterexample is sufficient to refute a false mathematical statement. Three participants, Sara, Laura, and Elaine (in the post-interviews) were aware of the fact that only one counterexample would be enough to refute a false mathematical statement; even though, they were not aware of the generality necessity of covering all possible cases when they constructed or evaluated a justification for true mathematical statements.

Although it should be cautioned to conclude that if PSTs failed to recognize that covering all cases is a necessary condition to validate a true statement, they would also fail to recognize that one counterexample would suffice to refute a statement, the majority of the PSTs, who were not aware of the generality rule of mathematical proofs, did not recognize that one counterexample was sufficient. Furthermore, some of these PSTs tended to believe that providing more than one counterexample would make the argument stronger. Simon and Blume (1996) show that many students think that giving one example is not enough to refute an argument. Similarly, Galbraith (1981) reported that only 18 percent of students believed that one counterexample was sufficient to disprove a statement. This study align with the results of those studies by documenting that the majority of the PSTs who did not demonstrate deductive reasoning believed that more than one counterexample would be needed to refute a statement and/or providing more than one counterexample would make the argument stronger.

Balacheff (1991) found that students relate to counter examples as bizarre instances and do not always recognize a counterexample as being sufficient to refute a universal statement. Similarly, Selden and Selden (1998) stated that students often fail to see a single counterexample as disproving a conjecture. They argue that this can result from perceiving a counterexample as the only one that exist rather than seeing it as generic. This study furthered the results of those studies by documenting that even though finding one counterexample was convincing for the participants to refute the statements, the participants attempted to provide more counterexamples when it came time to convince others, especially a teacher. Harel and Sowder (1998; 2007) define a proof as an argument that a person uses to convince himself or herself and others of the truth or falseness of a mathematical statement; thus, it consists of what constitutes "ascertaining" and "persuading" for a person or community. They use the terms ascertaining and persuading to describe complementary processes of proving as convincing oneself and convincing others. The results of this study also demonstrated that the ascertaining and persuading power of counterexamples differed.

There were different characteristics observed among PSTs when they were presented with false mathematical statements and/or when they attempted to refute the statements based on their conceptions of proof. The PSTs who demonstrated deductive reasoning attempted to test the conjectures by generating examples to check the validity of the statements presented. However, other participants who demonstrated various proof schemes other than deductive reasoning did not always need to generate examples to check the validity of the statements. Instead, they believed that the statements would hold true, thereby demonstrating various misconceptions underlying their decisions. The possible sources of difficulty in generating such examples were presumed to include the following: incomplete knowledge, inability to process existing knowledge, misconceptions, and insufficient logical knowledge (Zaslavsky & Peled, 1996). This study also documented that limited knowledge of perimeter and area concepts and the misconceptions that perimeter and area have a relationship influenced participants' example generation. Since they believed there was a relationship, they did not even attempt to test the method or to generate examples to prove that the method would hold true. However, the participants who demonstrated deductive reasoning generated examples to check the validity of the method. Bills et al (2006) argued that the process of exemplification is highly demanding, but yet has not been extensively investigated with regard to teachers. This study demonstrated that PSTs' attempt to generate examples indeed differed based on their conceptions of proof. Additionally, Rowland, Thwaites and Huckstep (2003) reported that desirable choice of examples depends on teacher's subject matter knowledge. This study aligned with the results of their study by documenting that PSTs who constructed counterexamples that were coded as irrelevant also demonstrated limited knowledge of the contents that underlined the tasks or questionnaire questions.

PSTs also demonstrated different characteristics when they attempted to generate counterexamples to refute false mathematical statements. While PSTs who developed deductive reasoning attempted to justify why and how their counterexamples would be appropriate and contradict the claims, PSTs who have not yet developed deductive reasoning only provided specific counterexamples without further explanation or justification. In other words, semi-general and general counterexamples were prevalent among PSTs who demonstrated deductive reasoning while specific counterexamples were prevalent among PSTs who did not demonstrate deductive reasoning.

Recommendations

As it has been argued throughout this study as well as in other studies, learning how to evaluate arguments and to construct an appropriate counterexample to refute false statements as discussed by one of the Common Core State Standards (2010) mathematical practices can only happen if students are provided ample opportunities to engage in tasks that not only require a true/false evaluation, but also a viable refutation to support their answer as part of their daily mathematical practices. However, this study along with other studies (e.g. Simon & Blume, 1996) documented the limited understanding of counterexamples among pre-service teachers who are just a step away from being a classroom teacher. In order for PSTs to be able to meet this requirement, they should have a solid understanding of how to refute false statements in mathematics. All these results highlight the need for attention to empowering instructions that are likely to lead PSTs to a better understanding of refutations. In addition to the importance of being attentive to conceptions as well as misconceptions of pre-service teachers regarding to constructing counterexamples, it is equally important to consider explanatory power of these counterexamples as facets of mathematical reasoning.

Examples are used as a communication device to explain thinking and reasoning in mathematics classrooms (Leinhardt 2001). Dahlberg and Housman (1997) documented the benefits of example generation to identify the correctness of conjectures and provide explanations. Thus, use of examples should be an important part of students' mathematics learning. As Dahlberg and Housman's study, this study also demonstrated that using examples is an effective strategy to evaluate the validity of mathematical statements. However, this study also demonstrated that the use of examples to identify the correctness of mathematical conjectures was not a common strategy among all participants, but only among students who demonstrated deductive reasoning. Therefore, I believe it could be a promising instructional strategy to encourage the use of examples strategy as inseparable part of our math classrooms.

Potari, Zachariades, and Zaslavsky (2010) argue that refuting conjectures and justifying invalid claims is a complex process that goes beyond the syntactic derivations of deductive proof. Understanding and mastering how to refute false conjectures and what is an appropriate counterexample requires the development of rationality and a specific state of knowledge. This study highlighted the importance of PSTs' understanding of underlying concepts when asked to evaluate conjectures and invalid claims and to construct refutations. In addition to PSTs' limited understanding of counterexamples, their limited understanding of the underlying concepts influenced their responses when it comes to refuting false conjectures. Therefore, more studies regarding to how pre-service teachers' content knowledge influence their evaluations and the process of refuting conjectures are needed.

Note

This study is part of the author' dissertation study, which was completed at Indiana University under the supervision of Dr. Enrique Galindo.

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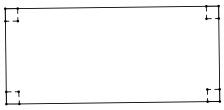
Appendix A - Sample Questionnaire Questions

Question1: Ms. Jones asked the following question to her 5th grade class:
"Possible or Not? A rectangle that is not a parallelogram. Justify your answer.
Katie claimed that the statement was possible and she provided the figure below to justify her answer.

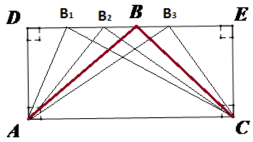


In your opinion, how would Ms. Jones mark Katie's justification? Please circle one of the options below and provide your reasons for choosing correct/incorrect.

Correct   Incorrect

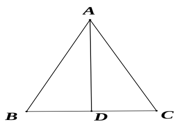
Your reason(s):

Question2: Kelly claimed that the area and perimeter of Δ ABC are going to be the same no matter where B is on segment DE of rectangle ADEC.



Do you agree with Kelly?
If you agree with Kelly: Please prove Kelly's conjecture.
If you disagree with Kelly: Please refute Kelly's conjecture.
If you partially agree with Kelly's conjecture: Please amend Kelly's conjecture and then provide a proof.

Question3: Jack was asked to justify that in an isosceles triangle ABC, the base angles (∠B and ∠C) are equal. He drew an angle bisector AD from ∠A and showed that AB ≅ AC, ∠BAD ≅∠CAD, and ∠B≅∠C. Therefore, Δ ABD≅ ΔACD by A.S.A. Hence, ∠B ≅∠C

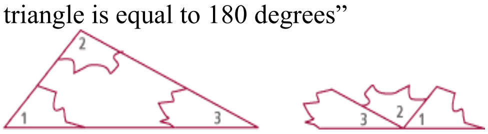


How would you mark Jack's argument? Please circle one of the options below and provide your reasons for choosing correct/incorrect.

Correct   Incorrect

Your reason(s):

Question4: What would be your reaction to the following justification of the sum of the interior angles of a triangle is 180 degrees:
"I tore up the angles of a triangle and put them together (as shown below), the angles came together as a straight line, which is 180 degrees. Therefore, the sum of the measures of the interior angles of a triangle is equal to 180 degrees"



Is this a proof? Why or Why not?