Representation in Teaching and Learning Mathematics

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Article Info	Abstract
	Representation is an important element for teaching and learning mathematics since utilization of multiple modes of representation would enhance teaching and learning mathematics. Representation is a sign or combination of signs, characters, diagram, objects, pictures, or graphs, which can be utilized in teaching and learning mathematics. Normally, there are four modes of representations in the domain of mathematics: (1) verbal, (2) graphic (3) algebraic, and (4) numeric. Certain type of representations can be dominant in teaching and learning mathematics; however, representation needs to be translated from one mode to another mode. Translation of modes of representation is an important skill that learners need to develop in order to be more proficient in learning mathematics. In the last couple of decades, the role of representation in mathematics education has been increased but requires more research studies to explore various aspects of representations.

Introduction

Representation is a crucial element for the teaching and learning of mathematics (Vergnaud, 1987). Duval (1995) states that "There's no knowledge that can be mobilised by an individual without a representation activity" (p. 15). The use of multiple representations in the teaching and learning of mathematics is a major topic in mathematics education that has gained significant importance in recent decades (Ozgun-Koca, 1998). For example, in geometry, mathematics teachers barely think of teaching geometry without using some kind of representations as pedagogical strategies. Kaput (1987) states that "representation and symbolization are the heart of the content of mathematics and are simultaneously at the heart of cognitions associated with mathematical activity" (p. 22). The role of representation in mathematics is further supported by National Council of Teachers of Mathematics (NCTM, 2000), which includes representation as one of the process standards in school mathematics curriculum. In fact, representation acts as a tool for manipulation and communication, and tools for conceptual understanding of mathematical ideas (Zazkis & Liljedahl, 2004). Representations also play essential role in teaching and learning of mathematics because they help teachers and students to grasp the abstract notion of mathematics (Roubicek, 2006). NCTM (2000) further explains the role and importance of the representations as follows:

> Representation should be treated as essential elements in supporting students' understanding of
> mathematical concepts and relationships; in communicating mathematical approaches, arguments, and

understanding to one's self and to others; and in applying mathematics into realistic problems situations through modeling (p.67).

Janvier (1987) states that use of representations in mathematical thinking is fundamental and most of the textbooks today make use of a wide variety of diagrams and pictures in order to promote mathematics understanding. Vergnaud (1987) further elaborate the role of representations as follows:

> Representation is a crucial element for a theory of mathematics teaching and learning, not only because the use of symbolic systems is so important in mathematics, the syntax and semantics of which are rich, varied, and universal but also for two strong reasons: (a) mathematics plays an essential part in conceptualizing the real world, (b) mathematics makes a wide use of homomorphism in which the reduction of structures to another is essential (p.227).

From a learning standpoint, students need to learn how to construct and interpret different forms of representational systems because they are essential tools for communication and reasoning about concept and information in mathematics (Greeno & Hall, 1997). Thus, teachers need to understand the effects of representations on students' learning in order to teach mathematics most effectively (Goldin & Shteingold, 2001). Vergnaud (1987) further affirms that representation is a crucial element for teaching and learning of mathematics for two strong epistemological reasons: mathematics plays an essential role in conceptualizing the real world and mathematics makes a wide use of homeomorphisms, in which reduction of structures to one another is essential. Regarding the importance of role of representations for the teaching and learning of mathematics, NCTM (2000) in the book "Principles and Standards for School Mathematics" states that: Instructional programs from prekindergarten through grade 12 should enable all students to: (a) create and use representations to organize, record, and communicate mathematical ideas; (b) select, apply, and translate among mathematical representations to solve problems; (c) use representations to model and interpret physical, social, and mathematical phenomena (p.67). Likewise, Dufour-Janver, Bednarz and Belanger (1987) identify several reasons for using representations in mathematics education and some of them are as follows:

- Representations are an inherent part of mathematics: There are topics in mathematics that are strongly associated with representation; someone cannot pretend to have studied these concepts without aid of representations. For example, the teaching and learning of concepts of function and cartesian graphics are always associated with representations.

- Representations are multiple concretizations of a concept: several different representations may encompass the same concept or the same mathematical structure. While presenting the mathematical concepts with the help of diverse representations, learners will grasp the common properties, which enable them to extract the intended concepts.

- Representations are used locally to mitigate certain difficulties: Mathematics textbooks and mathematic teachers make considerable use of representations during the teaching and learning process. When students struggle in learning certain concepts, teachers use several representations in order to ease the learning process for learners.

- Representations are intended to make mathematics more attractive and interesting: Authors use various types of representations quite extensively in recent textbooks to embellish the presentation of

mathematics to motivate the students. For example, varieties of representations can be seen in textbooks in order to present real world problems.

It is apparent from the related literature that representation has an important role in teaching and learning of mathematics. The importance is further supported by many scholars who agreed that representations plays an important role and its use is fundamental in teaching and learning mathematics (Aracavi, 2003; Goldin, 1987; Janvier, 1987; Kaput, 1987; Zazkis & Liljedahl, 2004; Zhang, 1997). Thus, many educators, psychologists, and researchers have defined, explained, and discussed the various aspects of the representations in relation to the teaching and learning of mathematics. In this regard, the paper aims to explore various aspects of representation in relation to teaching and learning mathematics. The paper is centered around in five different sections: (1) definition and meaning of representation, (2) types of representation, (3) modes of representation, (4) translation of representation, (5) implication of representation for teaching and learning mathematics, and (6) recommendation.

Definition and Meaning

The meaning and interpretation of representation is not uniform. Various types of definitions and interpretations are attributed to the notion of representation, particularly in teaching and learning mathematics (Zazkis & Liljedahl, 2004) because the meaning and interpretation of representation depends on the mathematical context (Mesquita, 1998). Moreover, the concept of representation is complex because it is not a static thing, but a dynamic process that is associated with an individual's mathematical thought process (Vergnaud, 1998). Goldin (1998) used the term an external representation; however, Lesh (1981) used the term only representation. In spite of some differences in defining and interpreting, most of the researchers describe about the representation in similar fashions, if not the same. Meanwhile, various distinctions have been made regarding the types, classifications, and nature of representation.

Duval (2006) states that representation is something that stands for something else. A representation is a sign or combination of signs, characters, objects, diagrams, or graphs, and it can be an actual physical product or mental process (Goldin, 2001). The mental process can be perceived as a mental image inside an individual's mind (head). It also refers both to process and product—in other words, to the act of capturing a mathematical concept or relationships in some form or to the form itself (NCTM, 2000, p. 67). In fact, representation may be a combination of something expressed on paper, existing in the form of physical objects, and constructed arrangement of ideas in one's mind (Janvier, 1987). Duval (2006) further states that "representations can also be signs and their complex associations, which are produced according to rules and which allow the description of a system, a process, a set of phenomena" (p.104). Thus, representations may denote and describe material objects, physical properties, actions and relationships, or objects that are much more abstract (Goldin, 1998, p. 4). Brinker (1996) defines representation, focusing on elementary mathematical concepts as follows:

> Representations refer to students' notations and pictures, already-made drawings such as pictures of portioned objects, and structured materials such as fraction strips and Cuisenaire rods. Structured in this case refers to materials that have been design for instruction of particular mathematical concepts (p. 1).

The fact is that Brinker's definition is more object-oriented and limited to only pictures and materials; does not include abstract idea inside of an individual's head. Goldin (2003) further defines representation as follows:

> A configuration of signs, characters, icons, or objects that can somehow stand for, or represent something else. According to the nature of the representation relationship, the term representation can be interpreted in the many ways, including the following: correspond to, denote, depict, embody, encode, evoke, label, mean, produce, refer to, suggest, or symbolize (p. 276).

Goldin particularly emphasizes the role of the configuration of signs, characters, icons, or objects in the representational system. He further contends that the notion of representational system is scarcely meaningful without the configurations of signs, icons, and so forth. According to Duval (1995), representation refers to a larger set of phenomena. For example, he states that figures can be a representation, but not every representation can be figure (for example, digits, symbols, diagram). He further states that the term image means the presentation of something through a reproduction relation, which can be physical objects or psychic construct. The role of representation (sign, characters, figures, icons etc.) and study of semiotics theory is important in mathematics education (Presmeg, 2014). Semiotics refers to the "the study or doctrine of signs" (Colapietro 1993, p. 179). Duval (1995) further states that the theory on registers of representation has three key properties: (I) there are many different semiotic representations of same mathematical objects, (II) each semiotic representation of the same mathematical object does not distinctly state the same properties of the object being represented, and (III) the content of semiotic representations must never be confused with the mathematical objects that these represent. Thus, representations include primarily various external forms of communication mathematical ideas such as diagram, signs, figures, characters, symbols etc., as well as internal structures formed inside an individual's head.

Types of Representation: External and Internal

Representation is categorized as internal and external based upon whether the representation is formed inside the mind of an individual as mental imagery, or expressed externally in the form of symbols, schemas, or graphs (Janvier, 1987). Various researchers discussed about the distinction between external and internal representations (Goldin & Shteingold, 2001; Goldin, 2003; Zhang, 1997). Zhang (1997) defines "external representations as the knowledge and structure in the environment, as physical symbols, objects, or dimensions, and as external rules, constraints, or relations embedded in physical configurations (p. 180)". In fact, external representation is designed to demonstrate and communicate mathematical relationships visually such as number lines, diagrams, algebraic equation, and so forth (Goldin & Shteingold, 2001). The external representation is the actual physical product produced either by a teacher or a student that can be used directly to teach mathematical ideas. For instance, a number line, an algebraic equation, or a triangle produced by students on a paper is called an external representation; teachers can point to such representation in a classroom and discuss their meanings to clarify the mathematical ideas and concepts (Goldin & Shteingold, 2001). Some external systems of representation are mainly notational and formal, while others show relationships visually or graphically (Godino & Font, 2010). Normally, notational and formal representation refers to algebraic expression, system of numeration, derivates, programming languages and so forth, whereas other representation denotes relationship visually or graphically such as number lines, graphs etc.

Unlike the external representation, internal representation is not a physical object; instead, it is a mental image which can be perceived as mental imagery inside the head of an individual. In doing so, the knowledge is stored in the form of structures, productions, and neural networks or in other forms which are collectively called mental imagery (Zhang, 1997). Cuoco (2001) affirms that:

> External representations are the representations we can easily communicate to other people; they are the marks on the paper, the drawings, the geometry sketches, and the equations. Internal representations are the images we create in our minds for mathematical objects and processes–these are much harder to describe (p. x).

In fact, the external and internal representations are intertwined with each other because external representation is the embodiment of a learner's internal perception or construct (Lesh, Post, & Behr, 1987). Zhang (1997) contends that external representation can be transformed into internal representation by memorization, and internal representation can also be transformed into external representation by externalization. However, in doing so, external representations are not necessarily identical to what goes into an individual's mind (Haciomeroglu, Aspinwall, & Presmeg, 2010).

There are controversies about the existence of internal representations because many scholars do not believe in the existence of the internal representation and even if it exists, it is almost impossible to investigate. For example, Lesh, Post, and Behr (1987) state that the external and internal dichotomy is artificial; only external representation is an observable activity. Similarly, Haciomeroglu, Aspinwall, & Persmeg (2010) also argue that internal representation is not necessarily identical to the external and it is difficult to interpret what goes inside individuals' mind. In fact, internal representation is not the carbon copy of external representation systems (Goldin, 2003). Because of the fact that the internal representation is not easily observable and there is controversy about its existence, discussion would be based primarily on external representation.

Modes of Representation

Mode of representation refers to a type of representation which is dominant while presenting a specific mathematical idea (Presmeg, 1986). External representation is classified into various categories based on attributes and the nature of representation employed in teaching and learning mathematics. Janvier (1987) proposes four modes of representation: verbal descriptive, table, graph, and formula (equational). Text, symbols, and sentences are ingredients of the verbal descriptive representation, whereas table is dominant mode in tabular representation. Similarly, drawings, pictures etc., are the main component of graphic representation and algebraic formula, equation etc., are the major means of expressing mathematical ideas in equational representation. Whereas, Lesh, Post, and Behr (1987) suggest five modes of representation (a) real scripts model, (b) manipulative, (c) static figural, (d) spoken language, and (e) written symbol. The script is experienced-based in which knowledge is organized around the real-world example that serves as general contexts for interpreting and solving other kinds of problem situations. In manipulative representation, elements such as arithmetic bars, base ten blocks, etc., have little meaning intrinsically, but the built-in relationships and operations fit many everyday situations. The static figural model includes different types of pictures or diagrams, which can be internalized as images during the teaching and learning of mathematics. The spoken

languages include specialized languages and sublanguages related to domains like logic, reasoning, etc. The written symbols refer to varieties of mathematical symbols and equations, specialized sentences and phrases, as well as normal English sentences and phrases.

Palmer (1978) proposes a different view about the representation. He contends that representational systems involve two related but functionally separate entities. The two related entities are representing world and represented world. The function of representing world is to reflect some or all aspects of the represented world in some fashion. In the represented-representing framework, the represented world can be modeled by representing world. In doing so, however, every characteristic of the represented world will not necessarily be presented by representing world. As an illustration, an example is provided as shown in Figure 1. In this example, the four rectangles in part A are the represented world and part B, C, D, and E are the representing world. Each vertical line with a different height in B is representing each rectangle of the represented world of part A. In this case, the longer the rectangle is, the taller the lines are. However, between A and C, the wider the rectangle is in A, the taller the line is in C. As shown in the example, there must be some specific relationship or correspondence between represented and representing world. There are certain entities of the represented and representing world. In fact, all of these representations in the figure are not the same. They contain some information that reflects some information about the world they represent.
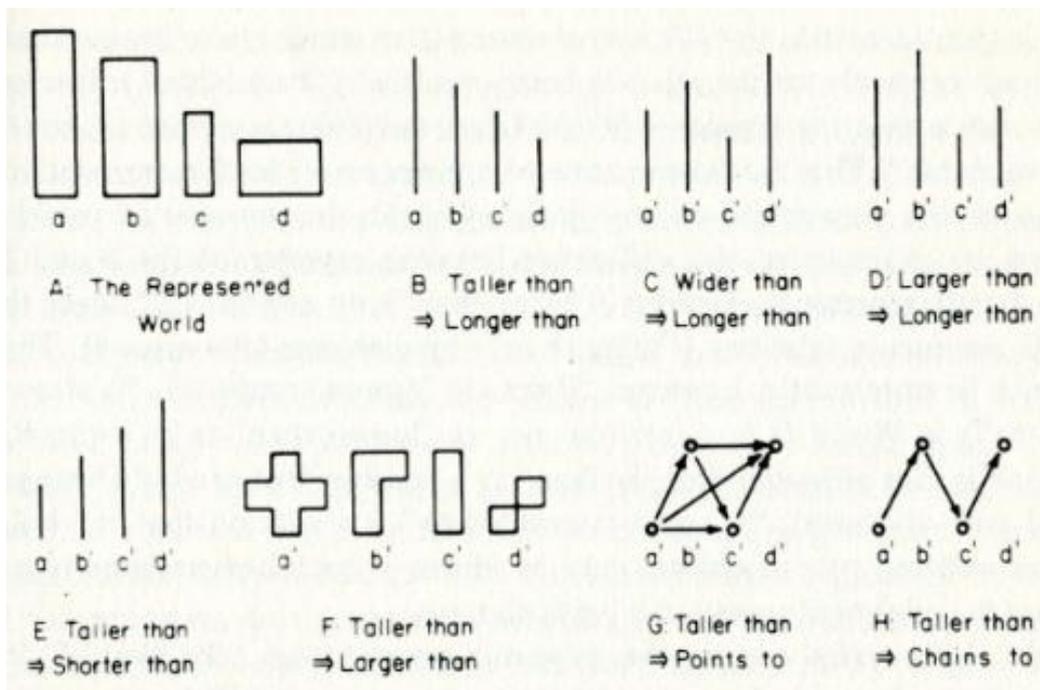

Figure 1. Example of Represented-representing World [From *Cognition and Categorization* by Palmer; E. Rosch, B. B. Lloyd, (Eds), 1978, p.263, Copyright, 1978 by Lawrence Erlbaum Associates.]

According to Palmer, the represented- representing world-framework has five entities: (a) the represented world, (b) the representing world, (c) what aspects of the represented world are being modeled, (d) what aspects of the representing world are doing the modeling, and (e) the correspondence between the two worlds. Palmer further contends that a representation is really a representation system if it includes all five aspects. The two worlds,

represented and representing, consist of objects that are characterized by certain relationships that hold among them. In fact, the function of the representing world is to preserve information about the represented world as precisely as possible. Palmer further states that there exists a correspondent (mapping) from objects in the represented world to objects in the representing world where at least some relationships in the represented world are structurally preserved in the representing world. For example, a world *X* is a representation of another world *Y,* if at least some of the relations for objects of *X* are preserved by relations for corresponding objects of *Y*.

Based on the information that can be provided by the representing-represented model of the same set object, Palmer explains two types of representation: informationally equivalent representation and nonequivalent representation. The former representation is concerned with preserving information and information consists of relations. Thus, informationally equivalent representations provide exactly the same information, regardless of the pair of representing world are different (B, E, F, G and H, in Figure 1). However, in nonequivalent representations, two representations that reflect different relations of the same objects are not equivalent in the sense that they do not preserve the same information (B, C, and D, in fig1). For example, in nonequivalent representations, one could not answer the same questions about the represented objects from both representations.

Kaput (1987) argues that in many cases one or both of the two worlds, represented and representing, may be hypothetical entities or even abstractions. Following the represented-representing framework, Kaput classifies the representation system into four broad and general categories: (a) cognitive and perceptual representation, (b) explanatory representation involving models, (c) representation within the mathematics, and (d) external symbolic representation. He further explains the different types of representations in mathematics. Some of the common representations that Kaput explains include morphisms, generic algebraic constructions, canonical building-block constructions (external), canonical building-block constructions (internal), approximation, feature/property isolation, and logic models. The different types of representation that Kaput describes are more focused on representation of abstract mathematics. Additionally, his classification is oriented to representation of one mathematical concept to another one by some sort of mathematical mappings.

However, Duval (2006) states that there could be enormous gap between representation world and represented world. He elaborates the gap by giving example of number representations. In the decimal notation "10" stands for the quasi-material representation "*IIIIIIIIII*" of the number "ten" gives the meaning. However, in this example it does not require understanding the way in which the used representation system function. Thus, Duval emphasize in the process of semiotic system of representations because mathematical processing always involves substituting some semiotic representation for another. For example, a specific object is associated with its representation, we don't know whether the representation refers to a specific object or to a general concept appears (D'Amore, 2005).

Miura (2001) classifies the representational system based on classroom activities. He states that there are two types of representations: instructional representation and cognitive representation. The instructional representation includes definitions, examples, and models that are used by teachers to impart knowledge to

students during teaching mathematics. By contrast, the cognitive representation refers to representations that are constructed by students as they try to make sense of mathematical concepts or attempt to find the solution to mathematical tasks. Whereas, Larkin and Simon (1987) describe only two types of external representation: sentential and diagrammatic. The sentential representation refers to the expression of problems with the help of sentences. In sentential representation, the natural language which is employed to express problems will be translated into simple formal sentences. In diagrammatic representation, diagrams will be used, and expressions require describing the components of the diagrams. Furthermore, they state that diagrammatic representation preserves the information about topological and geometric relations among the components of the problem, while sentential representation does not.

Wadsworth (2004) suggests different types of representations proposed by Jean Piaget, an influential developmental psychologist, particularly for the children of preoperational thought stage. The different representations include deferred imitation, symbolic play, drawing, mental imagery, and spoken languages. However, according to Piaget (1951), generally there are only two types of representation: symbols (pictures, tally marks etc.,) and signs (spoken words, written language, numerals, etc.) that play a dominant role in the learning process. Similarly, Bruner (1966) distinguishes the three modes of mental representations of knowledge: enactive, iconic, and symbolic. Enactive refers to learning by action, iconic indicates learning by visualization and summarizing pictures and images, and symbolic representation denotes learning by symbols and abstract mathematical language. Bruner states that these mental representations grow in sequence in learning individuals. From Bruner's standpoint, a student must first get the chance to actively work with concrete objects before making steps towards visual and symbolic representations.

Vergnaud (1998) proposes a representation theory in mathematics education, emphasizing the importance of signifier and signified. He contends that representation is a dynamic process involving different factors, and the organization of action has a significant role in the representation process. Thus, he opposes the metaphor of triangle as shown in Figure 2 because the metaphor of triangle is too static and does not offer any insight for the representation of relationships.

the thing

the representation of the thing   the symbol associated with the thing
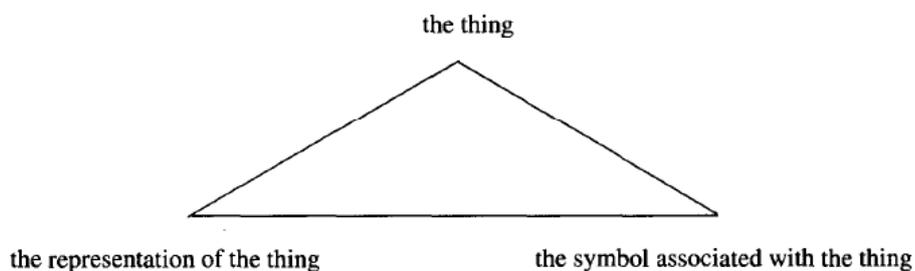
Figure 2. Metaphor of the Triangle [From "A comprehensive Theory of Representation for Mathematical Education", by G, Vergnaud, *The Journal of Mathematical Behavior,* 17, p.168. Copyright by the *Journal of Mathematical Behavior*, 1998.]

Based on both action and language, Vergnaud introduces the concept of a "scheme" as the invariant organization of behavior for certain situations, and proposes theorem-in-action and concept-in-action as

operational invariants, which are essential components for the scheme. The scheme includes: (a) goals and anticipations, (b) rules of action, information seeking, and control, (3) operational invariants, and (4) possibilities of inference. Defining concepts-in-action and theorem-in-action, Vergnaud states:

> Concepts-in-action: In every action, we select a very small part of the information available. Nevertheless, we need a wide variety of categories for this selection to take place, if one takes the word "category" to figure the wide meaning of object, class, predicate, condition, etc. Concepts-in-action are or, relevant, or not relevant or more or less relevant, to identifying and selecting information… There is no meaning in saying that the concepts of triangle, or number, or symmetry or scalar operator, or transformation are, in themselves, true or false; and still these concepts are relevant mathematical concepts to characterizing representation and action in mathematical task. Theorem -in-action can be true or false. This is a strong property, as it offers the only possibility of making more concrete the idea of computability and computable representation… Representation enable us to anticipate future events, and to generate behavior to reach some positive effect or avoid some negative one (Vergnaud, 1998, p. 173).

Vergnaud further introduces the concepts of reality associated with some real objects (situations) and representation related to the sentences and texts (schemes). He argues that concepts-in-action and concepts-in-theorem help to establish the connection between knowledge-in-action (objects) and knowledge-in-text (texts). However, in the process of transformation of knowledge from situations to knowledge to texts, there are important gaps between what is represented in the individual's mind and the usual meaning of words. He further contends that one cannot just consider that operational invariants are the same things as the signified of language or any other semiotic system. He proposes the following model in this regard as shown in Figure 3.
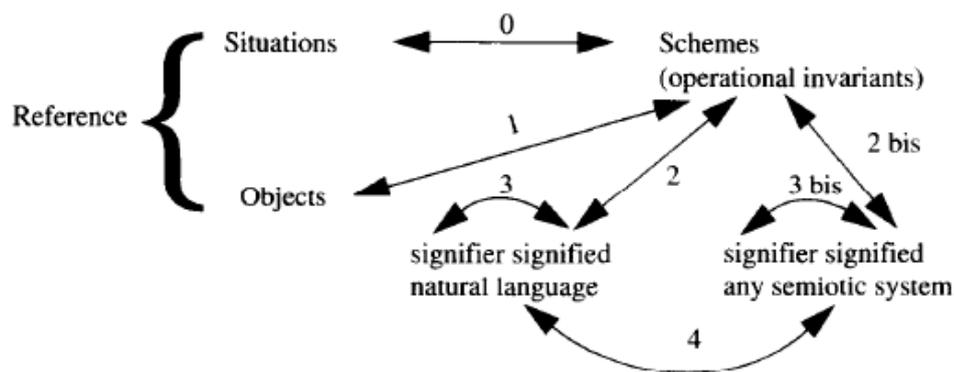
Arrow 0: the relationship between situations and schemes is the first source of representation, and therefore conceptualization.

Arrow 1: the formation of concepts implies the identification of objects, with their properties, relationships, and transformations. This is the main function of operational invariants, which are essential components of schemes.

Arrows 3 and 3bis: a language is a system of signifier/signified; a semiotic system also . The relationship between signifier and signified is not usually a one-to one correspondence.

Arrows 2 and 2bis: the relationship between operational invariants and the signified side of a particular linguistic or semiotic instance is not a one-to one correspondence either.

Arrow 4: natural language is a metalanguage for all semiotic systems. Again there is no one-to-one correspondence.

Figure 3. Alternative to the Triangle [From "A comprehensive Theory of Representation for Mathematical Education", by G, Vergnaud, *The Journal of Mathematical Behavior,* 17, p. 177. Copyright by the *Journal of Mathematical Behavior*, 1998.]

Janvier (1987) interprets representation in three different ways:

(a) representation refers to some material organization of symbols such as diagrams, graphs, schema etc., which denotes to other entities or modalizes various mental process,

(b) it implies a certain organization of knowledge in the human mental system or in the long term memory, and

(c) it also refers to a mental image.

The mental image that Janvier refers resemble with internal representation, which is not easy to track as what goes inside of an individual's head.

Godino, Batanero, and Font (2007) explain the relationship between objects in terms of onto-semiotic function. In the onto-semiotic function approach, a semiotic function is conceived, interpreting the ideas as the correspondence between antecedent (expression or signifier) and consequent (content or signified) established by subject according to certain criteria. They further contend that in onto-semiotic function, the role of representation is not completely undertaken by language (oral, written, graphics etc.). Furthermore, the actual content of the semiotic functions could be a personal or institutional, unitary or systemic, ostensive or non-ostensive object (Godino & Font, 2010). Ostensive objects refer to symbols, graphs etc., and non-ostensive objects refers to mathematical practices. Ostensive objects (e.g., symbols and graphs) and non-ostensive objects which learners bring in mind (which we bring to mind when doing mathematical tasks) that are textually, orally, graphically or even gesturally represented. Whereas, Saussure (1959) explains the concept of sign based on the structural theory of linguistics. He suggested that sign as the relation of signified and signifier, and the relation is inseparably one, where we cannot have one without the other. For example, when we discuss about the concept of a triangle, the sound image or impression forms in our head about the triangle (signifier) and the meaning associated with "sound triangle"- three-sided closed figure (signified) are inseparable. Thus, sign is the dualistic function between signifier and signified. Fried (2008) further elaborates the notions of synchronicity and diachronicity in explaining the process involved in teaching and learning mathematics. The synchronic view refers to a snapshot in time, while a diachronic analysis is a longitudinal one. Presmeg (2014) states that what is taught and learned in the given situation (synchrony) and the process involved as students engage over time with mathematical objects (diachrony) and the sign as a function play an important role in standing mathematical objects.

Goldin (1987) states that representation systems consist of a collection of elements called characters or signs. He describes the cognitive representation system in conjunction with mathematical problem solving, where the higher-level structure and language is associated with the representational system. The higher-level structures or languages include rules for forming configurations of configurations, networks of configurations, relation on the configurations, rules for assigning values to configurations, and operations on the collection of configurations. The configuration is the set of words, characters, or symbols. Goldin (1998) proposes a model for competence in mathematical problem solving based on four higher level languages: A verbal /syntactic system, nonverbal system for imagistic, formal notation system of representation, planning language. An affective system which monitors and evaluates problem solving progress. The main feature of this model is shown in Figure 4.
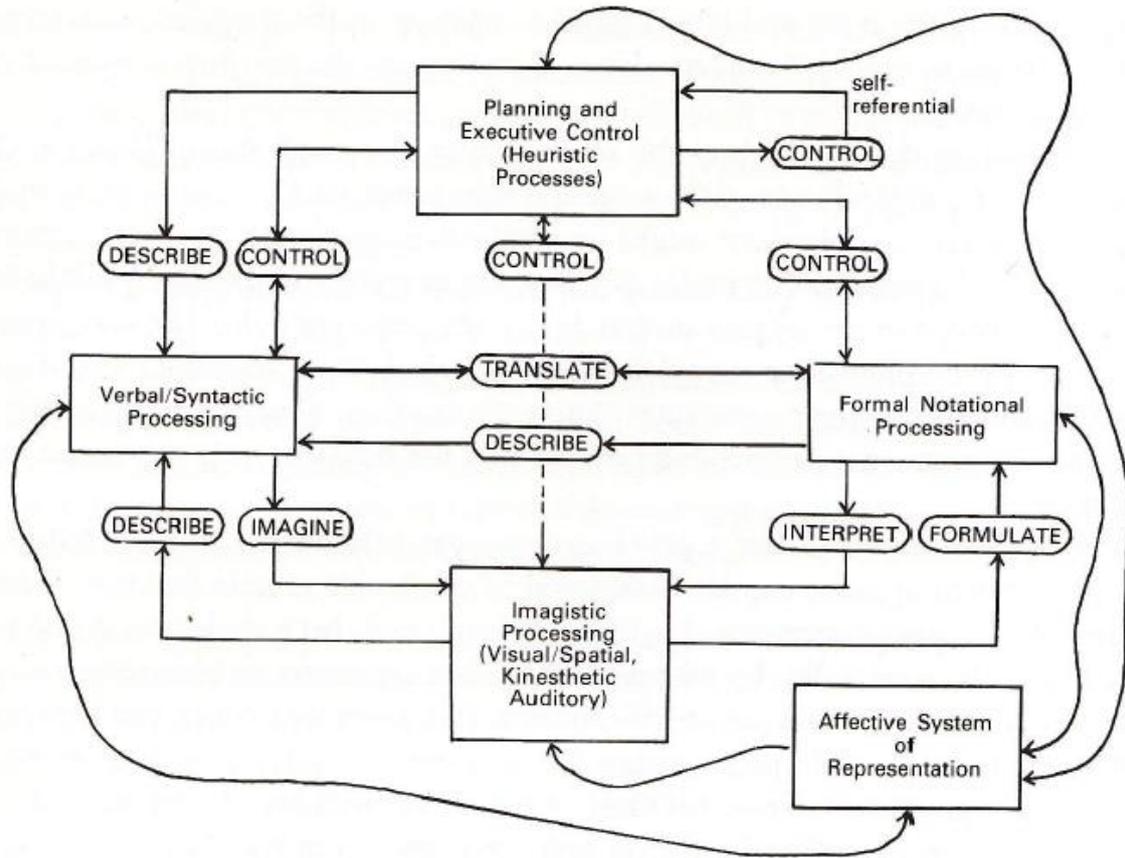
Figure 4. A Model for Competency in Mathematical Problem Solving [From "A comprehensive Theory of Representation for Mathematical Education" by G, Vergnaud, *The Journal of Mathematical Behavior,* 17, p. 136. Copyright by the *Journal of Mathematical Behavior*, 1998.]

In the model, we can see five representational systems. A verbal/syntactic system of representation can be described by means of signs, which are words and punctuation marks, together with correspondence between written and spoken words, rules for tagging by parts of speech and grammatical rules for combining words. An imagistic system of representation includes visual-spatial, kinesthetic, and auditory systems in order to be able to describe problem solving competencies. Formal notational system includes the ability to use the notations conventionally described as the language of mathematics and it also includes knowledge of how to represent problem states and make moves from one state to another in non-standard problems. For example, it includes numeration, algebraic notations, and so forth, and rules for manipulating them. The planning and executive control includes four dimensions with respect to which sub process is involved in their use. Planning and executive control representational system guides problem solving, including strategic thinking, heuristics and metacognitive capabilities. The affective representational system indicates the states of feeling that a problem solver experiences and expresses while solving a problem.

A representation, so called *rule of the three,* includes three modes of representations: symbolic, graphic, and numeric. Normally, all types of mathematical ideas and concepts, particularly in calculus, can be presented with the help of these three types of representation (Gleason & Hallett, 1992). The rule of three, however, becomes a rule of four. According to the rule of four, mathematical contents can be presented or expressed in four modes of

representation: graphical, numerical, algebraical, and verbal. These four modes of representation actually emerge out based on various ideas proposed by scholars in the domain of representations.

Moreover, these four modes of representations are most common and widely used in the field of teaching and learning mathematics. The graphic representation includes pictures, diagrams, coordinate planes, and other figural representations. The numeric representation refers to displaying data or mathematical ideas and concepts in an organized fashion, possibly in an ordered list or in a table. The algebraic representation indicates the use of symbol, formula, etc. The verbal representation includes written and spoken languages.

Describing these four modes of representations, only one representation may not be enough in order to present a mathematical idea; rather, more than one representation may be required at the same time. In doing so, one or the other kind of representation will have a dominant role to present mathematical ideas, concepts, or problems. For example, algebraic representation has a dominant role while presenting a quadratic equation; but verbal representation may be required at the same time. Depending on the nature of mathematics, one mode of representations likely has dominant role compare to another mode of representations and vice-versa. Additionally, students might have preferences of using one mode of representation over the other. Moreover, the different modes of representations need to be translated constantly from one mode of representation to another mode of representation in teaching and learning mathematics. Thus, translation of representations has a greater implication in teaching and learning mathematics.

Translation of Representations

In teaching and learning mathematics, different modes of representations are equally used and have important roles in understanding mathematical concepts. Both in teaching and learning mathematics, we constantly need to switch from one mode of representation to another mode of representation. The fact is that representation is certainty essential in learning mathematics and translation of representation is as important as utilizing the representation in teaching and learning mathematics (Duval, 2006). For example, students constantly change the employing one mode of representation to another while solving mathematics problems.

Using different types of representation often illuminates different aspects of complex mathematical ideas or relationships (NCTM, 2000). It is important to develop skills in students where they can translate one mode of representation to another based on the nature and situation of mathematics tasks. Following the work of Behr, Lesh, Post, & Wachsmuth, Lesh, Post and Behr (1987) state that translations (dis)abilities are significant factors influencing and problem-solving performance, and that fortifying and remediating these abilities facilitates the acquisition and use of elementary mathematical ideas. Thus, translation among the representations and transferring within them is an important process (Lesh, Post, & Behr, 1987) for effective teaching and learning of mathematics.

Janvier (1987) states that translation ability refers to psychological involvement going from one mode of representation to another mode of representation. Most researchers agree that translation is an important process

of successful use of representation (Dufour-Janvier, Bednarz, & Belanger, 1987; Janvier, 1987; Lesh, Post, & Behr, 1987a, 1987b) because translation of one mode of representation to another will provide flexibility in solving mathematics problems. Thus, one of the important goals of teaching mathematics is to teach students to translate one mode of representation to another without falling into contradictions (Hitt, 1998). In fact, the instructional strategies should include translation of all modes of representation because each representation has its own characteristics and poses different challenges for students doing mathematics problems (Gagatsis & Shiakalli, 2004).

Lesh, Post and Behr (1987), as aforementioned, classified the representation into five categories. The translation process takes place between and among these five types of representation as shown in Figure 5. We can see in Figure 5 that there are various arrows which indicate translation between and among these five modes of representations.
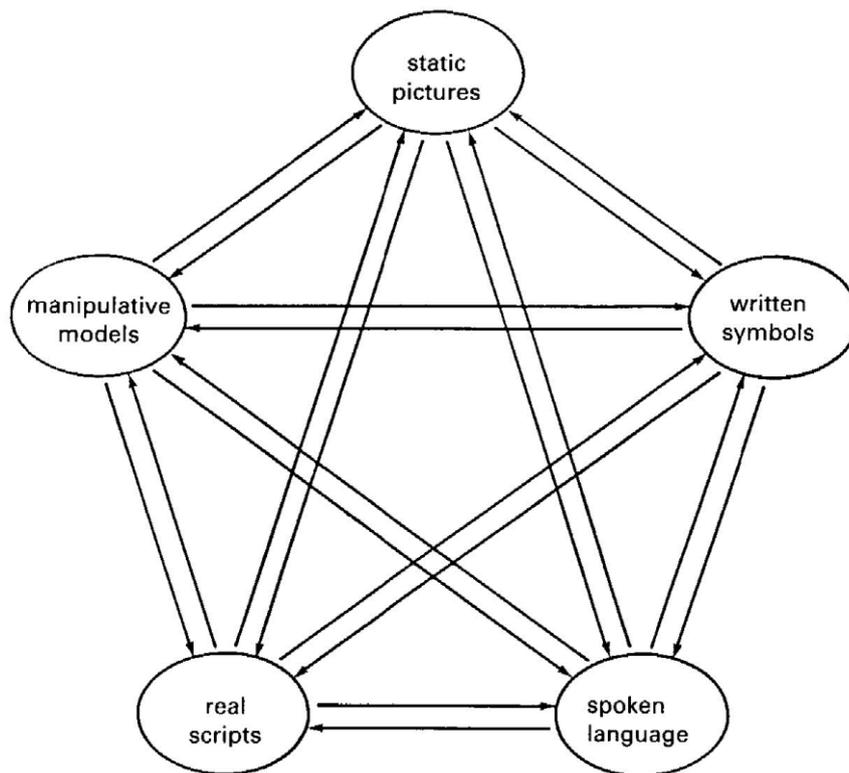

Figure 5. Types of Representation and Translation among Them [From *Problem of Representation in the Teaching and Learning of Mathematics* by R. Lesh, T. Post and M. Behr, 1987, p.34, Copyright, 1987 by Lawrence Erlbaum Associates.]

Lesh, Post and Behr suggest that representation tends to be plural in terms of translation processes in that solutions are often characterized by several partial mapping from parts of provided situations to parts of various representational modes. For example, a student may begin a solution by translating to one representational mode and may then map from this mode to yet another mode of representation as shown in Figure 6.
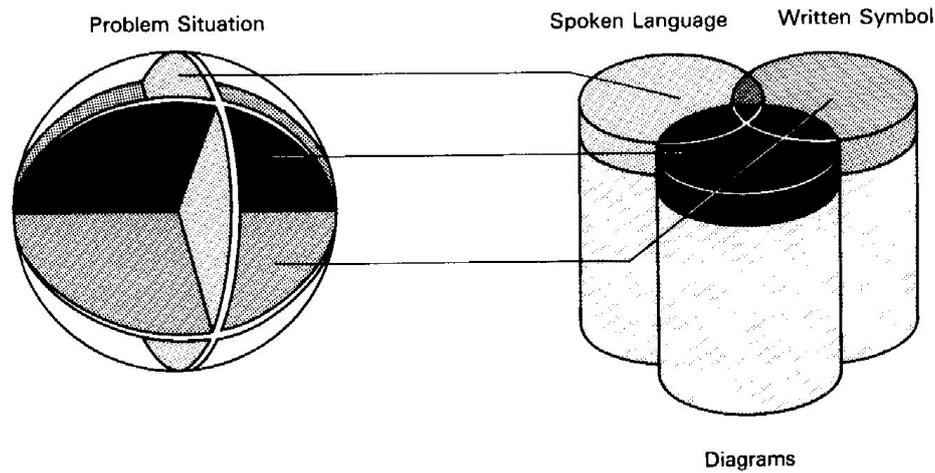
Figure 6. The Act of Representation [From *Problem of Representation in the Teaching and Learning of Mathematics* by R. Lesh, T. Post and M.Behr , 1987, p.38, Copyright, 1987 by Lawrence Erlbaum Associates.]

Janvier (1987a) describes the translation process among four modes of representations as shown in Figure 7. In the figure, we can see that translations between and among the several of modes of representations. For example, verbal descriptive representation can be translated into table and graph respectively by measuring and sketching. Similarly, graphic representation can be translated into verbal descriptive by interpreting the information that is given into graphic representation.

TRANSLATION PROCESSES

To From	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter Recognition	Computing	Sketching	

Figure 7. Translation Process among Four Modes of Representations [From *Problem of Representation in the Teaching and Learning of Mathematics* by C. Janvier, 1987, p.28, Copyright, 1987 by Lawrence Erlbaum Associates.]

Janvier further states that translation process involves two modes of representation of what he is called source-target paradigm framework. For example, the modes of equation and graph translations occur often between graphs to equations and equations to graphs. In the source-target parading translation framework, one has to

transform the source "target-wise" to look at it from a target point of view and derive the results. He contends that the translation process can be best developed in (symmetric) pairs, such as graph to verbal description (interpretation) and verbal to graphic (sketching). He added a few arrows to account for the alternative ways to achieve the translations. For example, we can see the translation that occurs from table to graph and similarly from a graph to formula as shown in Figure 8.

TRANSLATION PROCESSES

To From	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description		Measuring	Sketching	Modelling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter Recognition	Computing	Sketching	

Figure 8. Translation Process among Four Modes of Representations [From *Problem of Representation in the Teaching and Learning of Mathematics* by C. Janvier, 1987, p.28, Copyright, 1987 by Lawrence Erlbaum Associates.]

Janvier (1987b) widens the concepts of translation by using *schematization* (an illustration), which describes that translations between schematization is performed within a representation where translation would go from one point to another. He describes that this type of representation would be a sort of star like iceberg as shown in Figure 9. The translation process would consist in going from one point to another.
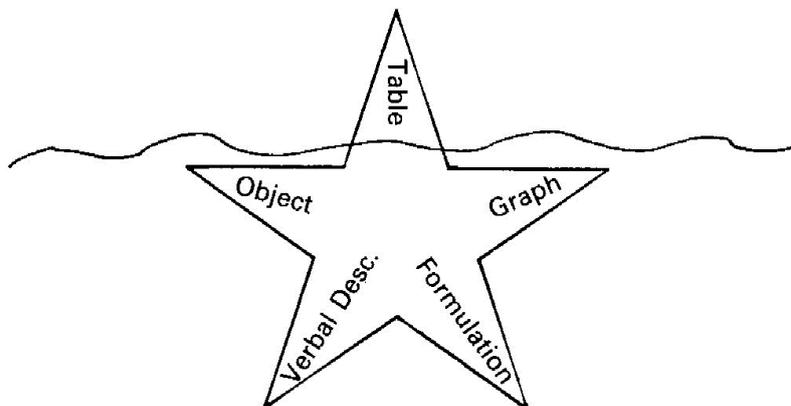

Figure 9. Representation as a Star-shaped Iceberg [From *Problem of Representation in the Teaching and Learning of Mathematics* by C. Janvier, 1987, p.69, Copyright, 1987 by Lawrence Erlbaum Associates.]

Janvier states that translation involves two modes of representations: the source (initial representation) and the target (final representation). For example, students may translate verbal descriptive representation to graphic representation, or vice versa. Janvier further states that translation ability can be best developed when students are asked to translate both from the source to the target and from the target to the sources. Consider the following verbal descriptive representation of a mathematics problem:

> From a ship on the sea at night, the captain can see three lighthouses and can measure the angles between them. If the caption knows the positions of the light houses from a map, can the caption determine the position of the ship? (NCTM, 2000, p. 69)

This problem can be translated into graphic representation. In graphic representation the ship and the light house become points in the plane and to solve the problem students do not need to know about the ship and the lighthouse. However, it may not be necessary to translate this problem in graphic mode of representation, but students need to learn to translate one representation to another, which provides them flexibility in doing and understanding ideas and concepts in an effective way. Furthermore, students who are more visual learners definitely tend to translate the problem in graphic representation in order to solve the problem. However, not necessarily all students need to translate the problem into graphic representation since students have different preferences for solution strategies. Having the skill of translating mode of representation would provide greater flexibility in solving mathematics tasks. For example, whether an algebraic or graphic representation is used; students might need verbal descriptive representation in order to present mathematical ideas or concepts in a clear and coherent way. Duval (1995), however, explains the two types of transformation based on the comprehension of the theory on registers of representation. The two types of transformations of semiotic representations that are radically different: treatment and conversions. Treatments are transformations of representations that happen within the same register: for example, carrying out a calculation while remaining strictly in the same notation system for representing the numbers, whereas Conversions are transformations of representation that consist of changing a register without changing the objects being denoted: for example, passing from the algebraic notation for an equation to its graphic representation or moving from a verbal statement into an algebraic operation, or draw the curve of a second-degree equation (Pino-Fan, et al., 2015). Despite some differences, most researchers contend that being able to translate between different modes of representation will result in a more in-depth understanding of mathematics (De Jong & van Joolingen, 1998; Lesh et al., 1987).

Implications for Teaching

Janvier (1987) states that use of symbolism in mathematics thinking is fundamental and most of the textbooks today make use of a wide variety of diagrams and pictures in order to promote understanding. Similarly, Goldin and Shteingold (2001) contend that teachers need to understand the effects of representation on students' learning in order to teach mathematics most effectively. As the different types of learning theories have developed, they unfolded that the roles and importance of representations for teaching and learning of mathematics have been increasing. Asli (1998) explains about reasons why students prefer to use representation while attempting mathematical problems. The fact is that acquiring new mathematical knowledge is

fundamental cognitive process, where the association between the representations and the object itself, the various signs and the designated things, works and its models etc., have crucial role (Duval, 2017). There are internal and external effects of using representations. The internal effects include personal preferences, previous knowledge and experiences, beliefs about mathematics, and rote learning; while external effects include presentation of problem, sequential mathematic curriculum.

In teaching and learning mathematics, students need early exposure to the various representations and lots of practice making translations between the representations (Ballard, 2000). Students need to learn how to construct and interpret different modes of representational systems because they are essential tools for communication and reasoning about concept and information in mathematics (Grenoo & Hall, 1997). However, mathematics teachers are unaware of using different modes of representations in appropriate in their instructional strategies. For example, Ma (1999) reported that many U.S. elementary school teachers lacked knowledge of representations in the teaching of division of fractions concept. We can find various textbooks, lesson materials, and other resources which include graphic mode of representations to illustrate the geometrical concepts and ideas. The developments of only one-sided preferences to utilize mode of representation result in narrow mathematical development for students because they do not have an opportunity to see mathematics problems from the other perspective. In fact, students who use only one modes of representation to solve mathematics problems might have limited understanding. For example, Ballard (2000) reported that students attempted to solve a problem based on probability with a tree diagram; however, tree diagrams were not an appropriate graphical method for representing the problems. Thus, the instructional strategies need to focus in students' development of utilizing different modes of representations. The fact is that students should be able to utilize various modes of representation in order to be more proficient in mathematics since some mathematics problems, for example, can be solved in an easier way using certain modes of representation than the others (Mainali, 2019). The instructional strategies also need to focus where students would be able to get chances to translate mathematics problem from one mode of representation to other mode based on their preferences for solution strategies. Mainali (2014) reported that when students utilized certain mode of representation to solve geometry problems, the majority of them had incorrect answers. However, students who employed different modes of representation, the majority of them were able to get the correct answer. Thus, based on the nature of mathematics problems, one specific mode of representation is not always useful to solve problems. Thus, it is equally important to infuse different types of modes of representation in teaching learning mathematics. Relying on only one mode of representation in instructional strategies, teachers inhibit students' opportunity learning mathematics employing different modes of representations. Thus, it is suggested that instructional strategies should be focused on incorporating different modes of representation in order to meet the student's preferences for solution strategies since some students like to use graph while other tend to use equation or some other numeric mode.

Recommendations

Several research studies reported that representations can be powerful tools to students learning (Bruner, 1966; Clements, 1999; Cuoco & Curcio, 2001; Fennema & Franke, 1992; Goldin & Shteingold, 2001; Greeno & Hall,

1997; Kilpatrick, Swafford, & Findell, 2001). There are significant numbers of studies which describe and discuss about the representation; however, surprisingly, there are meager number of researches which explain about the choice of representations for the purpose of teaching and learning of mathematics. The fact is that the different types of representations within a system are richly related to each other (Goldin & Shteingold, 2001), which need to be explored more in conjunction to teaching and learning mathematics. Additionally, more research studies need to be done in regard to the translation of representations. Janvier (1987a) says that the modes of representations and translation process can be best developed in (symmetric) pairs. Janvier (1987a) reported that graphic and verbal descriptive representations are the symmetric pair of representation. The Soviet researcher Kabanova-Meller states, "Mastery of geometric theorems is characteristically accomplished through the perception of diagrams (graphic representation) and intimately connect with the development of spatial images" (1970, p.7). The appropriate utilization of representations in teaching mathematics is important factor in student learning. How teachers decide to choose certain representations and how they use them in mathematics lesson activities? Which representation are more useful based on the nature of mathematics content and why? How teachers become proficient at using (not using) different modes of representations? Thus, more research studies needed to be conducted in order to investigate various aspects of representational system.

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