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Muhammad Qadeer Haider
Southern Methodist University, USA

Christine Andrews-Larson
Florida State University, USA

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Examining Learning Outcomes of Inquiry-Oriented Instruction in Introductory Linear Algebra Classes

Muhammad Qadeer Haider, Christine Andrews-Larson

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Abstract

This study evaluates the effectiveness of inquiry-oriented instruction in introductory linear algebra classes by comparing the performance of students who learned introductory linear algebra concepts in inquiry-oriented settings (TIMES students) with the students who attended other linear algebra classes (Non-TIMES students). We used the assessment data from 461 students (271 TIMES and 190 Non-TIMES students) which were collected from 19 linear algebra classes at 15 institutes across the country. The linear algebra assessment was given as a post-test to all students, and TIMES students performed significantly better than Non-TIMES students. Overall, the difference in the performance of both groups was statistically significant in the entire assessment, procedural subscale, and conceptual scale of the assessment. In a pair-wise comparison of TIMES and Non-TIMES classes, only those TIMES classes performed significantly better than the Non-TIMES classes on the linear algebra assessment where instructors have more experience with inquiry-oriented teaching.

Introduction

Active learning has shown promising results in improving student performance in undergraduate Science, Technology, Engineering, and Mathematics (STEM) courses, and many researchers are working to improve teaching and learning experience in STEM courses through some form of active learning (Freeman et al., 2014). Promoting student-centered instruction and shifting from traditional teaching is part of current research efforts to enhance the learning experience in undergraduate STEM classes. In active learning classes, instead of just listening and taking notes, students read, write, discuss, and engage in problem-solving and higher order thinking tasks (Bonwell & Eison, 1991). Active learning is a broad concept, and many types of non-traditional teaching can be considered active teaching, such as problem-based learning, inquiry-based learning, project-based learning, and inquiry-oriented learning. In contrast with active learning classes, in traditional lecture classes, students passively listen to an expert, take notes, and ask unprompted questions occasionally (Bligh 2000, Freeman et al., 2014). Mataka & Taibu (2020) reported that students who used inquiry module and involve in explicit discussion of concepts and misconception showed better conceptual understanding than their peers in the control group. Inquiry-oriented teaching is one of the many types of active learning, and that is the focus of this study. Different studies have reported that the inquiry-oriented teaching has shown positive results on student learning and retention of knowledge as compared to traditional lecturing (Freeman et al., 2014; Laursen et al., 2014; Kwonet al., 2005).

Although inquiry-oriented teaching has shown promising results in some studies (Bouhjar et al., 2017; Kwon et al., 2005), the shift from traditional to student-centered teaching is a challenging task (Johnson & Larsen, 2012; Wagner et al., 2007). Therefore, researchers are working to support instructors who are interested in changing their teaching from traditional to more inquiry-oriented style teaching. Some researchers have developed specific curricular materials to support inquiry-oriented teaching (e.g., Larsen et al., 2013; Rasmussen & Kwon, 2007; Wawro et al., 2013) and others have developed support models to help instructors for inquiry-oriented teaching using specific inquiry-oriented teaching materials (Johnson et al., 2015).

Teaching Inquiry-Oriented Mathematics: Establishing Supports (TIMES) is a research project funded by the National Science Foundation (#1431393, 1431595, 1341641). TIMES provides model for helping instructors implement inquiry-oriented teaching. The TIMES instructional support model has three main components; curricular support material, a summer workshop for instructors, and weekly online instructors' work groups (Johnson et al., 2015). The project supported three focus areas in undergraduate mathematics namely, abstract algebra, differential equations, and linear algebra; this paper focuses on instructors supported in implementing Inquiry-Oriented Linear Algebra (IOLA) teaching materials. IOLA teaching materials include student and teacher materials to support the inquiry-oriented teaching of linear algebra concepts (http://iola.math.vt.edu). The student-facing IOLA materials consist of challenging task sequences which lead to formal linear algebra concepts. In this paper, we focus on IOLA materials related to four focal topics of linear algebra: span and linear independence, systems of linear equations, linear transformations, and eigenvalues and eigenvectors.

The purpose of this paper is to examine student learning in the context of inquiry-oriented teaching in introductory linear algebra classes. Specifically, we focus on identifying the impact of inquiry-oriented classroom instruction on students' conceptual and procedural understandings of linear algebra concepts. In the rest of the paper, we refer the instructors and their students who participated in inquiry-oriented teaching and learning through the TIMES project as TIMES instructors and TIMES students, respectively. Similarly, we use the terms Non-TIMES instructors and Non-TIMES students for non-participating comparison instructors and their students respectively. In this paper, we compare the performance of TIMES students with the performance of Non-TIMES students using a valid and reliable linear algebra assessment. We also look at students' performance within their groups. In this study, we address the following two specific research questions:

1. How does students' performance (overall, within TIMES, and within Non-TIMES) on conceptual items compare to students' performance on procedural items?
2. Do TIMES students outperform Non-TIMES students on the overall assessment?
    a. Do TIMES students outperform Non-TIMES students on the procedural subscale?
    b. Do TIMES students outperform Non-TIMES students on the conceptual subscale?

In this article, we first briefly discuss the research theorizing conceptual and procedural understanding and inquiry-oriented teaching. We then discuss the inquiry-oriented teaching materials used by linear algebra instructors in the context of the TIMES project and describe the TIMES project more broadly to give the background of the study. Later, we report the details of participants and data sources, methods of analysis, and findings of the study. Lastly, we discuss the role and implications of inquiry-oriented teaching in undergraduate mathematics courses.

Conceptual and Procedural Knowledge

Mathematics education literature emphasizes the conceptual understanding of mathematics and disconnection between mathematical concepts and procedures in school and college math classes. Conceptual understanding is often characterized by connections among different ideas concepts, whereas procedural understanding tends to be thought of in terms of following specific procedures and applying formulas to solve problems. Skemp (1976) used the terms of instrumental learning and relational learning of mathematics. Instrumental mathematics learning refers to the application of fixed plans to find the answers to the questions (sometimes described as "rules without reasons" (p. 2). Simplistic interpretations equate procedural understanding with instrumental learning; we agree with Star's (2007) argument that procedural understandings are ideally deep, flexible, and critically applied.

Hiebert and Lefevre (1986) defined conceptual knowledge as: "rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3). This is distinct from procedural knowledge, which is "rules or procedures for solving mathematical problems. It is also a familiarity with the individual symbols system and with the syntactic conventions for acceptable configurations of symbols (Hiebert & Lefevre, 1986, p. 7)." NCTM Principles and Standards for School Mathematics (2000) emphasize the importance of conceptual understanding of mathematics among elementary and secondary students, and we argue the focus is equally important for college math courses. Understanding mathematical concepts are critical in advanced mathematics but not trivial (Melhuish, 2015).

Without conceptual understanding to accompany it, procedural understanding of mathematical ideas may only enable students to follow a problem-solving procedure efficiently and accurately; conceptual understanding is needed for a student to use this knowledge in practical life or a different situation. Hiebert (2013) argued that conceptual understanding enables a student to apply mathematics in real-world situations and use acquired mathematical knowledge in a new situation. Therefore, learning math without conceptual understanding is unlikely to prepare students for higher level mathematics. Moreover, students may not be able to utilize their mathematical knowledge in their practical life completely. For example, if students only learn computational skills associated linear algebra topics – like row reduction, and solving system of equations, etc., – there is no guarantee that students also understand the concepts behind those computations and how to use these concepts to propose solutions to a given real-world problem. However, we also cannot ignore the importance of computation in learning of mathematics, and argue that the dichotomy between procedural and computational understanding is false. Both types of knowledge can enforce each other for comprehensive learning of various mathematical concepts (Star, 2005).

Casual use of terms conceptual knowledge and procedural knowledge has created some misconceptions, and one may perceive procedural knowledge as a rote memorization of steps instead of another type of knowledge. Conceptual knowledge is considered the deep understanding of concepts and procedural knowledge is taken as superficial knowledge of symbols manipulation (Star 2005, 2007). Following Star (2013), we use the terms procedural and conceptual knowledge to refer to two types of knowledge, not to suggest that one type of knowledge is more valuable than the other.

In light of the conceptual and procedural knowledge discussion above, we argue that conceptual knowledge and procedural knowledge are the two sides of a coin which cannot be separated but each side has its own identity and value, and there should be a clear connection between procedural and conceptual knowledge. Procedural knowledge alone, without the underpinning of the concepts, may be of limited use. On the other hand, procedural knowledge can play a supporting role in conceptual understanding for students (Star, 2005).

Inquiry-Oriented Teaching

Inquiry-oriented teaching is a particular way of supporting active learning in which both teachers and students participate in learning through inquiry (Kwon et al., 2005). In inquiry-oriented classrooms, students inquire into the new mathematical concepts, and teachers inquire into students' ways of reasoning and learning of new mathematical concepts to adjust their instruction and formalize mathematical ideas in a way that relates to and builds on students' current ways of reasoning about a mathematical concept (Rasmussen & Kwon, 2007). In inquiry-oriented classrooms, the student engages in mathematical discussions with their classmates, by asking students to propose and defend conjectures, and solve new problems without receiving step-by-step help from their teachers.

Kuster et al. (2017) identify four instructional components of inquiry-oriented teaching: generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation. Kwon et al. (2005) argue that students' active verbal involvement in class discussion and instructional material – inspired by the instructional design theory of Realistic Mathematics Education (RME) – are two central features of inquiry-oriented classes. The RME design heuristic of guided reinvention aims to leverage students' informal and intuitive ideas to develop more formal mathematical language and notations (Rasmussen & Keynes, 2003; Rasmussen et al., 2005).

Inquiry-Oriented Teaching Material

Researchers have developed curricular material to support inquiry-oriented teaching for different undergraduate courses like abstract algebra, differential equations, and linear algebra (Larsen et al., 2013; Rasmussen & Kwon, 2007; Wawro, Rasmussen, Zandieh, & Larson, 2013). These teaching materials consist of research-based sequences of problem-solving tasks which were developed through iterative cycles of research and design. The research-based activities were developed based on the design research approach of classroom teaching experiments which is a cyclic process of task design and ongoing analysis (Cobb, 2000). In the design research cycle, the research team designs and pilot tests an initial task sequence; analysis of the pilot data informs revisions and adjustments. Paired and classroom teaching experiments, and analysis of data from these experiments further inform revisions to the sequence of tasks. Implementation of these sequences of tasks is generative for student learning; instructors pose probing questions to students, and student contributions generate opportunities for productive class discussions that build toward important mathematical ideas. Inquiry-oriented instructional materials are intended to help educators to connect students' informal ideas with more formal and conventional understandings (e.g., Wawro et al., 2012).

Teaching Inquiry-Oriented Mathematics Establishing Support (TIMES)

The inquiry-oriented curricular materials are tools intended to help teachers, but inquiry-oriented teaching is not a simple and straight-forward task. Researchers have reported some of the challenges in student-centered teaching like such as time management, content coverage, student resistance, and departmental disagreement on specific teaching interventions (e.g., Anderson, 2002; Johnson & Larsen, 2012; Speer & Wagner, 2009). To address these challenges, the TIMES project provided participating instructors with three main supports: curricular materials, summer workshops, and weekly online instructor workgroups (Johnson et al., 2015).

The curricular materials provided to instructors participating in the TIMES project provide a specification of learning goals, the rationale for every task, notes for implementation of tasks, and some examples of students' work as a reference for teachers. TIMES instructors participated in a three-day summer workshop intended to introduce the inquiry-oriented curricular material and to develop a shared and clear understanding of inquiry-oriented instruction. The weekly online instructors' work groups met once every week for an hour to discuss and work on the selected lessons of the curricular materials. The entire group also watches and discusses a selected clip of implementation of task for every instructor. The purpose of this activity was to help instructors interpret and respond to student thinking in a way which supports student learning.

Method

Data Collection

Throughout three academic years (2014-2017), the TIMES research team collected a large amount of data for different purposes. We collected pre- and post- interviews with participating instructors, classroom videos of selected lessons (3-4 hours of instruction per instructor), and students' work on a common post-assessment from TIMES and Non-TIMES classes. However, the scope of this study is restricted to the analysis of students' assessment data – which was collected during the academic years of 2015-2017 – to compare the performance of TIMES and Non-TIMES students.

The linear algebra assessment was administered as a paper-and-pencil test, in TIMES and Non-TIMES classes, towards the end of the term of instruction. The assessment consists of nine questions with multiple choice and open-ended response subparts. Students were given up to one hour to complete the assessment. Overall, we collected the assessment work of 461 students, which we used in the analysis of this study. The data were collected from 19 linear algebra classes taught by 19 different instructors at 15 different institutes across the country in two academic years. Across the two academic years, we collected the assessment data of 271 TIMES students from 12 TIMES classes and 190 Non-TIMES students from six classes.

Linear Algebra Assessment

The assessment of linear algebra concepts aligns with the four focal topics included in IOLA curricular materials: span and linear independence, systems of linear equations, linear transformations, and eigenvalues and eigenvectors. Our team reviewed the literature on teaching and learning of linear algebra, commonly used linear algebra textbooks, and also consulted three external content experts to craft the first draft of the assessment. The team worked to identify and develop questions that can reveal students' conceptual understanding of linear algebra concepts (in addition to somewhat more typical problems that emphasize procedural skills).

The finalized version of the assessment carries nine questions, where some questions are open-ended, some are multiple-choice, and other questions in the assessment are a combination of multiple-choice and open-ended items. For some of the multiple-choice questions (MCQs) students have to pick only one of the given options, and for other MCQs, students must choose more than one correct choice. The open-ended questions of the assessment are free-response items where students either need to explain their choices on the previous MCQ or write the answer to a question in their own words. A sample of each type of MCQs and open-ended questions are shown in Figure 1.

In the development of linear algebra assessment, we focused on aligning test items and the topics test intended to measure (establishing content validity) and ensuring test items measure what they are purported to measure and do not measure irrelevant content (establishing construct validity). The content validity was established through expert validation to ensure items had potential to measure students' conceptual understanding of focal topics. The construct validity was ensured by conducting and analyzing clinical interviews of eight university undergraduate students. Additional details on the linear algebra assessment development and validation are available in another paper (Haider, 2018; Haider, 2019).

To gauge the difference in the performance of students on procedural and conceptual subscales separately, we categorized every item on the test as a procedural or conceptual item. We followed Hiebert and Lefevre (1986) definition of procedural and conceptual understanding of mathematical concepts to group items in procedural subscale and conceptual subscale. Procedural knowledge is dealing with formulas and procedures to solve a given problem, but conceptual knowledge is about making the connection among different concepts for a complete understanding of a topic (Hiebert & Lefevre, 1986). A summary of all question format and their categorization of procedural, conceptual, or mixed on the linear algebra assessment is available in Table 1.

1. Answer the following questions regarding the set of vectors $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

   a. Which of the following best describes the <u>span</u> of the set V?

   - ☐ i: A point
   - ☐ ii: Two points
   - ☐ iii: A line
   - ☐ iv: Two lines
   - ☐ v: A plane
   - ☐ vi: Two planes
   - ☐ vii: A 3-dimensional space

   b. Justify your response to part a.

   c. Which of the following are in the <u>span</u> of $V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$? (Check all that apply.)

   - ☐ i: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$
   - ☐ ii: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
   - ☐ iii: $\begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}$
   - ☐ iv: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
   - ☐ v: $3.1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$
   - ☐ vi: <u>any</u> vector in $\mathbb{R}^3$

   d. Explain in general how you can determine if a given vector is in the span of some other set of vectors.

Figure 1. Example of Multiple-Choice Questions and Open-Ended Questions

If students can use an existing formula or a set of steps to answer a question, the question was categorized as a procedural item. For example, we classified part c of the question 1 (see Figure 1) as a procedural item because a student can take scalar multiples or linear combinations of the given vectors to see which of the given options is in the span of set V. On the other hand, if no existing formula or set of steps can help student to answer the question, we concluded that it requires conceptual understanding to answer; such items were categorized as conceptual. For example, we categorized part a and part b of question 1 (see Figure 1) as conceptual items because the student need to connect the idea of span (all possible linear combinations of the pair of vectors given) with the idea the geometric interpretation that two non-zero vectors three-dimensional space that don't point in the same direction will span a plane. Because students must connect these ideas, we categorized these items as conceptual.

We also developed a scoring criterion, based on the item format, to score the assessment data. Items with one correct choice counted as one point; and items with multiple correct options had one point possible for each option assigned based on whether that option was answered correctly or not. All open-ended questions were scored on a three-point scale, with individual rubrics for each item. All dichotomous items were assigned one point for correct response and zero otherwise. The set of items classified as procedural and conceptual were each validated quantitatively as subscales by using Cronbach alpha as an indicator of self-consistency of procedural and conceptual subscales. The value of Cronbach alpha is .79 for conceptual and mixed items and .70. All details of scoring criteria and validation are available somewhere else (Haider, 2018).

Table 1. Categorization of Procedural, Conceptual, or Mixed on the Linear Algebra Assessment

Question/ Testlet	Topics	Item/ part	Subscale/ Category	Possible Score	*Question Type/format
1	Span and Linear Combination of Vectors	a	Conceptual	1	MCQ (Pick one)
		b	Conceptual	3	Open-Ended MCQ (Pick All that Apply)
		c	Procedural	6	
		d	Mixed: P&C	3	Open-Ended
2	Linear Independence	a b	Conceptual	1	Circle One
			Conceptual	1	Circle One
		c	Conceptual	1	Fill in the Blank
3	Interpretation of RREF	a b	Mixed: P&C	1	MCQ (Pick one) Open-Ended
			Conceptual	3	
4	Product of Matrices	-	Mixed: P&C	5	MCQ (Pick All that Apply)
5	The system of Linear Equations	a b	Procedural	3	Open-Ended
			Procedural	3	Open-Ended
		c	Conceptual	1	Circle One
6	Linear Transformation	a b	Conceptual	1	MCQ (Pick one) Open-Ended
			Conceptual	3	
7	Invertible Matrices	-	Mixed: P & C	3	Open-Ended
8	Eigenvalues	-	Procedural	3	Open-Ended
9	Eigenvectors	a	Conceptual	6	MCQ (Pick All that Apply)
		b	Conceptual	3	Open-Ended

* MCQ (Multiple Choice Question)

** Mixed: P&C = A mixed item which is Procedural and Conceptual

Analysis

Using the scored assessment data, we conducted various analyses to compare the performance of students on the linear algebra assessment on the different scales. Apart from looking at the overall assessment score of all students, we compare students' scores within and across TIMES and Non-TIMES groups to gauge any difference in the performance of students on conceptual and procedural subscales. We consider the assessment data from TIMES classes independent from Non-TIMES classes. Since no pre-test data collected, we decided to use an independent sample t-test to see if the difference in scores of both groups is statistically meaningful for overall assessment. However, for procedural and conceptual subscales, we used paired samples t-test because mixed items were common in both subscales, which created dependency. We also compared the performance of seven TIMES classes with comparable Non-TIMES classes – wherever comparable data was available – using Multivariate Analysis of Variance (MANOVA) technique because we was looking at the difference in the performance on three continuous response variable total score, procedural score, and conceptual score.

Initially, we compared the performance of students on procedural and conceptual subscales. We pooled all students score, TIMES and Non-TIMES together, and compared students' score on procedural and conceptual subscale. Later, we compared TIMES and Non-TIMES students' performance on procedural and conceptual scale separately. We used paired-samples t-test to compare students' performance on procedural and conceptual subscales. The comparisons of students' performance suffice to answer the first research question on a comparison of students' performance (overall, within TIMES, and within Non-TIMES) on procedural and conceptual subscales.

In response to the second research question, we compared the performance of TIMES students against the performance of Non-TIMES students on three scales: overall assessment, procedural subscale, and conceptual subscale. First, we pooled all TIMES and Non-TIMES data and looked at the performance difference of both groups on three scales using independent t-test. Second, we conducted a pair-wise comparison of TIMES and Non-TIMES classes using MANOVA. For pair-wise comparison, we paired every TIMES class with a comparable Non-TIMES class, wherever data was available. In this portion of the paper, we will refer the comparison of comparable TIMES and Non-TIMES classes as a pair-wise comparison. The pair-wise comparison of both groups at class-level can reveal more variability in the performance of TIMES and Non-TIMES students.

Our research group preferred to collect data on treatment and control groups from the same institutes. However, finding both groups at the same institute was not possible in every case. In some cases this happened because some institutes are small and only offer linear algebra course once every year or two; in other cases, the same instructor teaches linear algebra every semester. Therefore, we were only able to collect TIMES and Non-TIMES data from same institutes in some cases; these are the first four pairs in Table 2. For other three pairs, we paired the TIMES and Non-TIMES institutions based on institutes' acceptance rate of new students, the total enrollment of the institutes, and the enrollment of linear algebra classes at both institutes. To pair two institutes, we divided all institutes into small (less than 5,000), medium (between 5000 and 15,000), and large (more than 15000) categories based on the enrollment size of every school. We paired TIMES and Non-TIMES small school together if the difference in the new student acceptance rate of both schools and the difference in the linear algebra class size was less than 20%. Based on the pairing criteria, we were able to pair T4 with nT8, T5 with nT6, and T9 with nT7 schools. We couldn't find comparable Non-TIMES schools for TIMES schools T11, T12, T13, T14, and T15.

In the available data set, the number of TIMES classes is more than the number of Non-TIMES classes, and for five TIMES classes, we could not find a suitable comparable institute to collect comparison data. Therefore, we have not included these five classes in the pair-wise comparison. However, we have added the data from these five classes in the overall comparison of TIMES and Non-TIMES students. We have also paired two TIMES sections with one Non-TIMES section because these three sections were available at one institute and classes were suitable for cleaner comparison, which are the asterisk entries in Table 2.

Table 2. Comparison of Paired Linear Algebra Classes for Pair-wise Comparison

	TIMES Classes				Paired Non-TIMES Classes			
Pair	School Code	Acceptance Rate	School Enrollment	Class Enrollment	School Code	Acceptance Rate	School Enrollment	Class Enrollment
1	T1	83%	42,477	37	nT1	83%	42,477	55
2	T2	61%	3,798	23	nT2	61%	3,798	25
3	T3*	11%	8,353	20	nT3*	11%	8,353	18
4	T3*	11%	8,353	25	nT3*	11%	8,353	18
5	T4	64%	6,741	17	nT8	70%	6,325	15
6	T5	62%	748	6	nT6	68%	904	6
7	T9	75%	21,093	48	nT7	58%	32,929	59
8	T11	76%	22,350	11				
9	T12	71%	2,144	12				
10	T13	74%	7,336	56				
11	T14	47%	4,883	14				
12	T15	52%	1,491	15				

We used paired-samples t-test to compare students' scores on the procedural subscale with their scores on the conceptual subscale for three groups: TIMES students, Non-TIMES students, and all students grouped together. Since mixed items were counted twice in procedural subscale and conceptual subscale which created inter-item dependency, a dependent t-test was used for this analysis. However, to compare the average scores of TIMES group with Non-TIMES group on overall test, procedural subscale, and conceptual subscales, we used an independent t-test as there was no dependency among both groups of students. For pair-wise comparison of seven pairs of TIMES and Non-TIMES classes, we used MANOVA to find any difference in means of both groups on three dependent variables (overall score, procedural score, and conceptual score) based on the independent variable TIMES/Non-TIMES and pairs.

Results and Discussion

This study has two goals: the first goal is to report the difference in the performance of students on procedural and conceptual questions on four focal topics of introductory linear algebra courses. Another goal of the study is to compare the performance of students in TIMES classes with the performance of students in Non-TIMES classes to report if inquiry-oriented teaching help more in student learning. Initially, we look at the performance of all students on the entire test, procedural subscale, and conceptual subscale categories. Later, we compare the difference in the performance of all TIMES students with Non-TIMES students. Finally, we look at the difference in the performance on TIMES and Non-TIMES classes by pairing each TIMES class with a comparable Non-TIMES class.

Performance of Students on Procedural and Conceptual Subscales

To investigate performance across students on procedural and conceptual subscales, we pooled the scored data of TIMES and Non-TIMES students. A substantial amount of pooled data can reveal if there is a statistically significant difference in the performance of students on both subscales. The paired t-test result shows that students' average score on procedural (including mixed items) and conceptual (including mixed items) subscales was significantly different. Students performed better on procedural questions – where problems can be solved using procedural manipulation rather than a conceptual understanding of underlying concepts – as compared to conceptual questions. Overall, students' average score on the procedural subscale was 77%, and on the conceptual subscale, the average was considerably lower, i.e., 64%. This difference between scores on the two subscales is also statistically strongly significant (see Table 3).

Within TIMES and Non-TIMES groups, the difference in performance on the two subscales aligned with the overall trend, i.e., students performed significantly better on procedural items than on conceptual items. The statistics showed that all students (TIMES and Non-TIMES) were more successful on procedural items rather than conceptual items and the difference in performance on both subscales was also statistically strongly significant for both groups. Within the TIMES group, the average score on procedural subscale was 79%, and on conceptual subscale, the average was 67% (see Table 4). Similarly, within Non-TIMES group, there was a statistically significant difference in the scores on procedural items (i.e., 73%) and conceptual items (i.e., 59%) (see Table 5).

Table 3. Performance of All Students on Procedural and Conceptual Items

Comparison	n	Percentage (Mean)	SD
Procedural (0-27)	461	77% (20.87)	4.53
Conceptual (0-36)	461	64% (23.07)	6.57

Note. t_{922}= 6.72, $p < .001$ (Using paired sample t-test)

Table 4. Performance of TIMES Students on Procedural and Conceptual Items

Comparison	n	M (Percentage)	SD
Procedural (0-27)	271	79% (21.47)	3.94
Conceptual (0-36)	271	67% (24.22)	6.12

Note. t_{540}= 6.60, $p < .001$ (Using paired sample t-test)

Table 5. Performance of Non-TIMES Students on Procedural and Conceptual Items

Comparison	n	M (Percentage)	SD
Procedural (0-27)	190	19.72 (73%)	5.08
Conceptual (0-36)	190	21.44 (59%)	6.86

Note. t_{378}= 3.05, $p < .005$ (Using paired samples t-test)

Comparison of Performance of TIMES and Non-TIMES Students

In comparing the performance of TIMES students with Non-TIMES students, we look at the difference of average score between the two groups on the entire test, conceptual subscale, and procedural subscale. Initially, we pooled all TIMES and Non-TIMES data and compared the performance of both groups, and we again used an appropriate t-test to check if the difference between the average score of the two groups is statistically meaningful on overall test and on both subscales. Later, we examine the performance of both groups at class-level by pairing comparable TIMES and Non-TIMES classes using MANOVA.

Comparison of Pooled TIMES and Non-TIMES Students

In the category of performance on the entire test, TIMES students performed significantly better than Non-TIMES students based on the average score of both groups. On average, TIMES students scored 72%, and Non-TIMES students scored 65% on the test (see Table 6). The difference between average scores of both groups was strongly significant ($t(460)=4.57$, $p < .001$). On the procedural and conceptual subscales, the overall performance of TIMES students remained significantly better than the performance of Non-TIMES students, as it was on the entire test. On the procedural subscale, the mean score of TIMES students was 79%, whereas the mean for Non-TIMES students was72%. Similarly, TIMES students scored 67% on the conceptual subscale and average of Non-TIMES students remained 59%, again this average difference between both groups on two subscales was statistically significant, as shown in Table 7 and Table 8.

Table 6. Performance of Students on the Entire Linear Algebra Assessment

Comparison	n	M (Percentage)	SD
TIMES	271	36.62 (72%)	7.88
Non-TIMES	190	33.00 (65%)	9.07

Note. t_{460}= 4.57, $p < .001$ (Using independent sample t-test) *(Maximum 27 points)*

Table 7. Performance of Students on the Procedural Subscale

Comparison	n	M (Percentage)	SD
TIMES	271	21.31 (79%)	3.94
Non-TIMES	190	19.57 (72%)	5.08

Note. t_{460}= 4.17, $p < .005$ (Using paired sample t-test) (Maximum 27 points)

Table 8. Performance of Students on the Conceptual Subscale

Comparison	n	M (Percentage)	SD
TIMES	271	24.22 (67%)	6.12
Non-TIMES	190	21.44 (59%)	6.86

Note. t_{460}= 4.10, $p < .001$ (Using paired sample t-test) *(Maximum 36 points)*

The comparison of students' scores on the test showed that overall, TIMES students outperformed Non-TIMES students in all three categories (entire test, procedural subscale, and conceptual subscale).

Pair-wise Comparison of TIMES and Non-TIMES Classes

In addition to the pooled comparisons of TIMES and Non-TIMES students in different categories, we also compared TIMES classes with comparable Non-TIMES classes, wherever data was available for comparable classes. The aggregated score of all TIMES and Non-TIMES students from different institutes may mask significant variability across different settings, while the class level comparison of both groups can reveal more variability in the performance of both groups. Therefore, we compared TIMES and Non-TIMES classes by pairing similar classes in both groups according to the criteria mentioned above. Again, every pair of classes was compared on the three categories, entire test, procedural subscale, and conceptual subscale.

We used two-way MANOVA to determine the effect of inquiry-oriented teaching over three dependent variables overall test score, score on procedural subscale, and score on conceptual subscale. To demonstrate the results, we included descriptive statistics and results from multivariate test to show the effect of independent variables (TIMES or Non-TIMES and Pairs) on the dependent variables (overall score, procedural score, and conceptual score). The descriptive statistics are given in Table 9, which shows the mean, standard deviation, and size of the class in each pair. The average score of TIMES students remains better or equal than the Non-TIMES students in every pair.

Table 9. Pair-wise Comparison of TIMES and Non-TIMES Classes at a Glance

			Overall (0-51)			Procedural (0-27)			Conceptual (0-24)		
Pair	Comparison	n	M	SD	P	M	SD	p	M	SD	p
1	TIMES	32	29.72	6.98	**.03**	18.06	3.92	**.03**	11.66	4.37	.17
	Non-TIMES	55	26.13	7.83		15.75	5.13		10.38	3.96	
2	TIMES	17	42.88	4.53	**.04**	23.88	1.65	.11	19	3.72	**.04**
	Non-TIMES	21	37.81	8.95		21.81	4.50		16	4.99	
3	TIMES	17	40.12	6.14	.55	23.12	3.08	.42	17	4.20	.77
	Non-TIMES	19	38.89	5.96		22.26	3.12		16.63	3.41	
4	TIMES	27	40.96	7.45	.32	23.33	3.50	.29	17.63	4.45	.42
	Non-TIMES	19	38.89	5.96		22.26	3.12		16.63	3.42	
5	TIMES	17	36.71	5.74	.14	21.35	2.71	**.03**	15.35	4.30	.70
	Non-TIMES	15	33.60	6.09		18.87	3.29		14.73	4.71	
6	TIMES	6	36.50	9.48	.71	21.17	4.31	.87	15.33	5.61	.48
	Non-TIMES	6	34.83	4.54		21.50	2.26		13.33	3.56	
7	TIMES	48	36.75	8.56	**.01**	20.71	4.39	.63	16.04	4.86	**<.001**
	Non-TIMES	59	32.75	8.12		20.28	4.50		12.46	4.36	

Note: The numbers in parenthesis show the range of possible score in each category. e.g., Overall (0-51) indicates the range of total possible score.

To satisfy MANOVA assumption of equal covariance in both groups, we used Box's M test. The Box's Test of Equality of Covariance produced M value of 39.62 with a p value of .05, which is a borderline case of non-significance. Therefore, we assumed that variance and covariance matrices of both groups are somewhat equal, for three dependent variables, which satisfy the assumption of MANOVA.

Through MANOVA test, we observed statistically significant difference between the performance of TIMES and Non-TIMES students within each pair on three dependent variables. From the Multivariate test (see Table 10), for independent variables Pairs and TIMES or Non-TIMES, p is less than < .05 from Wilks' Lambda test. Therefore, we rejected the null hypothesis and there is some statistical difference between the performance of TIMES and Non-TIMES students in each pair on three dependent variable total score, procedural score, and conceptual score together.

Table 10. Multivariate Test Results for Pairs, TIMES or Non-TIMES, and Intersection

Effect		Value	F	Hypot-hesis df	Error df	Sig.	Partial Eta Sq.
Pairs	Wilks' Lambda	.054	97.107	18.000	967.81	.000	.622
TIMES or Non-TIMES	Wilks' Lambda	.971	3.409	3.000	342.00	.018	.029

Note: Included only necessary statistics and removed rest of entries for space adjustment

To identify for which dependent variables there is a difference in the performance of TIMES and Non-TIMES students, we reported ANOVA results. First, we used Levene's Test for Equality of Variance to confirm the homogeneity of variance condition among three dependent variables. Levene's test results were statistically non-significant for the total score and conceptual score with p was greater than .05. Hence, non-significance of Levene's test showed the variance across the total scores and conceptual scores are same. Therefore, the assumption of equal variance at univariate level is satisfied. However, for the procedural score, the p-value was less than .05which can be because the procedural scale has fewer questions and there is an overlap of mixed items between procedural and conceptual subscales.

Test of Between-Subjects Effect (see Table 11) showed that there is a statistically significant difference of sources (Pairs and TIMES or Non-TIMES) on dependent variables(Total score, Procedural score, and Conceptual score).In order to identify the pairs where the difference in performance was statistically significant and where it was not, we used t-test and check the p-value for every pair, as already shown in Table 9 above. Among some pairs, the differences in the mean scores for some dependent variables were strongly significant – for example pair 1, 2, 5, and 7 – as the p-values of t-test remained less than 0.05. The classes in these pairs followed the broader trend that students in TIMES classes performed significantly better than students in Non-TIMES classes on the entire assessment. However, in other cases, the data did not follow the trend, where average score of TIMES classes was better than Non-TIMES classes, but the difference of means was not statistically significant for these pairs – for example in pair 3, 4, and 6. The non-significant but positive difference in the performance of TIMES and Non-TIMES students shows that it needs a larger sample size to determine if the positive difference among these pairs is also statistically meaningful.

Table 11. Tests of Between-Subjects Effects

	Dependent Variable	Type III Sum of Squares	df	Mean Square	F	Sig.
Pairs	Total Score	6951.94	6	1158.66	21.036	.000
	Procedural Score	1672.51	6	278.75	16.745	.000
	Conceptual Score	5029.20	6	838.20	36.450	.000
TIMES or Non-TIMES	Total Score	503.54	1	503.54	9.142	.003
	Procedural Score	92.54	1	92.54	5.559	.019
	Conceptual Score	232.76	1	232.76	10.122	.002

Note: Included only necessary statistics and removed the rest of entries for space adjustment

Students in TIMES classes of pair 1, 2, and 7 performed better than their counterparts on the overall assessment. In pairs 1 and 5, TIME students performed significantly better on procedural subscale than the Non-TIMES students. In pair 5, TIMES students' average score was still better than the Non-TIMES students on the entire test and conceptual subscale, but this difference was not statistically significant. Similarly, students in TIMES classes of pairs 2 and 7 performed significantly better than the students in Non-TIMES classes on the conceptual subscale, but the difference on the procedural subscale was not statistically significant.

In the results of the pair-wise comparison, it is worth mentioning that the pairs 1, 2, and 7 – in which students in TIMES classes performed significantly better than students in Non-TIMES classes – participating TIMES instructors had more than one year of working experience with IOLA material and inquiry-oriented teaching. On the other hand, instructors in pair 3, 4, and 5 used the IOLA material for the first time in their linear algebra classes. We neglected the pair 6 because the class size was too small for any statistically significant comparison. It needs further investigation to find the reason for the difference in performance on procedural subscale for pairs 1, 2 and difference in performance on conceptual subscale for pairs 2, 7.

The primary goal of the study was to compare the performance of students in inquiry-oriented linear algebra classes with the students in non-inquiry-oriented classes to see if students in inquiry-oriented classes gained a better conceptual understanding of introductory linear algebra concepts. The most consistent finding of the study is that overall students in TIMES classes performed significantly better in all categories than the students in Non-TIMES classes. In a pair-wise comparison of TIMES and Non-TIME classes, the average score of TIMES classes was always better than their counterparts, but this difference of means was not always statistically significant. The analysis also showed that students perform better on procedural questions than the conceptual questions.

Apart from an overall comparison of both groups, the pair-wise comparison also showed some promising results. Though students in TIMES classes have not significantly outperformed students in Non-TIMES classes in all paired comparisons, the average score of students in TIMES classes was always more than their counterparts. Additionally, in pairs 1, 2, and 7, where TIMES' students performed significantly better than Non-TIMES students, two of the TIMES instructors have used IOLA material for inquiry-oriented teaching in previous years, and one instructor had experience with the development of IOLA material. On the other hand, among all pairs where the difference of performance was not significant, the TIMES instructors were using IOLA material for the first time in their linear algebra classes. We conjecture that an instructor's experience with IOLA materials can help student learning in inquiry-oriented classes, but more targeted studies are needed to confirm the impact of involvement with IOLA material to improve student learning. The connection between instructor experience with IOLA material and students' performance is aligned with the findings of Andrews-Larson et al. (2019) that instructors consistently using IOLA material improve their eliciting and building on student contribution which can result in rich mathematical argumentation driven by students' contribution.

Conclusion

The findings of this study replicate some results of earlier work on the effectiveness of inquiry-oriented teaching on student learning. Kwon et al. (2005) reported that students in inquiry-oriented differential equations class showed better conceptual knowledge and equal procedural knowledge when compared with students in traditional differential equations classes. Additionally, students in inquiry-oriented differential equations class showed better retention of concepts than their counterparts. This study carries a larger sample size as compared to Kwon et al., (2005), but our findings are similar. We found that students gained better conceptual understanding in linear algebra classes which was evidenced by their performance on the conceptual subscale of the linear algebra assessment and students in TIMES classes performed equally or better on procedural subscale. However, it needs further investigations to gauge and report the difference in retention of linear algebra concepts in inquiry-oriented and traditional classes.

Further, emerging evidence suggests that, depending on implementation, outcomes in inquiry-oriented mathematics classes may differ by gender (see e.g., Laursen et al., 2014; Johnson et al., 2020). Our study did not disaggregate outcomes by gender, so we are unable to determine whether this trend is also present in our data. Others have examined ways in which particular instructional settings may shape students' disciplinary experiences in gendered ways (Schulte & Wegner, 2021). The analysis of Reinholz et al. (2021) indicates that instructional norms can function to include or exclude women in inquiry-oriented settings, and that this inclusion or exclusion is quantitatively linked to differences in students' learning outcomes.

Our findings suggest that the support model of the TIMES project – including summer workshops, IOLA curricular materials, and weekly online workgroups – helped instructors in implementing inquiry-oriented teaching (Fortune et al., 2019; Kuster et al., 2019) which ultimately helped their students in gaining a better understanding of linear algebra concepts. This study was not intended to pinpoint which component of the TIMES support played what role in students' success. Further investigations are needed to find out all the factors and the impact of each factor which helped students in better conceptual understanding in inquiry-oriented classes, and the ways in which these outcomes may vary for particular groups of students. This work will carry important implications for instructors learning to implement inquiry-oriented instruction in ways that support improved learning outcomes that are equitable across groups of students.

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Author Information

Muhammad Qadeer Haider
https://orcid.org/0000-0002-5047-3573
Southern Methodist University
USA

Christine Andrews-Larson
https://orcid.org/0000-0001-7331-0817
Florida State University
USA
Contact e-mail: *cjlarson@fsu.edu*