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Rezan Yilmaz[1], **Ziya Argun**[2]

[1] Ondokuz Mayıs University

[2] Gazi University

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Abstract

Mathematical abstraction is an important process in mathematical thinking. Also, visualization is a strong tool for searching mathematical problems, giving meaning to mathematical concepts and the relationships between them. In this paper, we aim to investigate the role of visualizations in mathematical abstraction through a case study on five pre-service secondary mathematics teachers. To do this, we formulated instructions that are appropriate for abstracting the concept of congruent figures or objects, and then determined the visualizations used by the participants, how they established these and what kind of visual images they had. The data were collected through class observations and semi-structured interviews. Data analysis showed that visualizations were used frequently by the participants and differently while engaged in abstraction. They used isomorphic and homeomorphic visualizations and these were mostly in coordination and internalization processes. The participants also produced different visual images which were commonly concrete and dynamic ones. The visualizations that were used in the abstraction processes significantly aided the understanding of the concept of congruent figures and strengthened the visual images of the participants.

Introduction

Abstraction is an important process in mathematical thinking. Numerous studies have been carried out to determine a definitive process for mathematical abstraction and specify suitable methods for it. However, a theoretical response to the issue of what types of information are necessary has not been found.

As we think or talk about mathematical objects or processes, we link these with perceptions in our minds. These perceptions are representations of the object or concept. Therefore, having rich representations that include a large number of related pictures of concepts increases success in mathematics. If a student improves his/her ability to consciously abstract from mathematical cases, one can say that he/she has developed advanced mathematical thinking. Abstraction is an appropriate construction process in the mind, involving specifying the relationships between mathematical objects and turning these relationships into specific expressions independent of the mathematical objects. Appropriate visuals help students engage actively in abstraction. These kinds of visuals also help students build mental representations and deductions (Tall, 1991).

Mathematics is a field that is done with describing and objectivizing the concepts abstracted from real cases, and most of the descriptions, which are regarded as meaningful according to experiences, emerge visually (Bishop, 1989). Visualization is a significant view in mathematical understanding and evaluation; because visualization advances these thought processes, it not only organizes data in the form of meaningful frameworks but is also an important factor in the sense that it guides the analytical development of solutions (Fischbein, 1987).

Visualization is commonly believed to play an important role in mathematics teaching. Ben-Chaim, Lapan, & Houang (1989) drew attention to the role of visualization in the development of inductive, deductive, and proportional reasoning. According to the authors, visualization is a central component of many processes and is essential to fulfilling the transformation of thinking from the concrete to the abstract. Wheatly (1991) inferred that images based on processes are crucial for using mathematical information. The advantage of visualization is that it develops the power of multidimensional thinking in individuals. In this sense, individuals develop the capacity for collective discussion and exchange of ideas by looking at events from different perspectives.

A limited number of studies on visualization in mathematics education have been conducted. Educational research has attached great importance to the role of visual images in solving mathematical problems (Stylianou, 2001). However, none of these aforementioned studies have examined abstraction. In Presmeg's (2006) visualization-related studies, she proposed that there is an urgent need for research on visualization not only in solving mathematical problems but also in learning and teaching mathematics at any level. She emphasized that analyzing visualizations is important in the development of abstraction. Consequently, it is believed that this study is of importance in terms of filling this void in mathematics education by projecting the place and way of visualization in the whole abstraction. Moreover, it is assumed that conducting this study with prospective teachers will contribute to educating teachers on using visualization in future.

In that context, the research problem of this study is investigating the role of visualization on the abstraction process. Therefore, 'congruence of two geometric objects' was evaluated as the conception. We sought answers to the sub-problems of the research, which ask participants what visualizations they use and how they benefit from them if they use any, and what type of visual images they had. We applied a case study on five pre-service secondary mathematics teachers. To this end, we designed some instructions that are appropriate for abstracting and then determined the visualizations that the participants used and visual images that they had.

Theoretical Framework

Abstraction

Abstraction is a constructional process, in which mental frameworks are established from mathematical frameworks and vice versa. It is a process of distinguishing an object's common property (or properties) from the object itself and naming this property. It occurs as an isolation of a concept from its specific quality, and the direction of the process goes from a context group to a concept (Sierpinska, 1994; Skemp, 1986; Tall, 1988). Furthermore, abstraction does not require the storage of all types of information. Only important and appropriate ones are stored, whereas the rest are discarded, with their representations converted into more abstract concepts (Hampton, 2003). Looking into the development of mathematical ideas reveals that the abstraction process requires deep conceptual organization. This act generally results in the formation of new objects. Such formation is a core characteristic of abstraction. Abstracting characteristics and processes continue with the use of new objects, which are then the bases of abstractions in subsequent stages (Ferrari, 2003).

The first fundamental studies on hierarchical abstraction theory were conducted by Skemp (1986). His investigations are an extension of the studies of Dienes (1961). According to Skemp, abstraction is a result of the abstracting process and provides the opportunity to realize new experiences, which have similarities that we classified beforehand. When abstracting is regarded as an activity and abstraction is the end-product (object) obtained as a result of this activity, the object obtained is called a concept. Two types of abstraction were identified by Piaget (1977): empiric and reflective abstraction.

Empiric abstraction is based on superficial similarities and requires information on everyday concepts. It is not only related to the awareness of similarities that stem from our experiences but also initiates generalizations from the specific to the common and from some to all, as well as the deductions of the common properties of objects. It focuses on descriptions that consist of the basis of general properties and advances teaching theories that can comprehensively improve understanding (Mitchelmore & White, 1995; Piaget & Garcia, 1989; Skemp, 1986; White & Mitchelmore, 2002).

According to Piaget (1977), reflection in reflective abstraction is based on the actions of individuals, and concepts and relationships are abstracted by bringing these actions together. Reflective abstraction is a structuring of logical-mathematical frameworks in the course of the cognitive development of an individual. It depends on the formation of concepts, as indicated in some theories, and serves to reveal these concepts through mental and systematic analysis with the help of the connections and relationships between objects. It has been viewed as a general coordination of actions and its source has been referred to internally and as a subject (Beth & Piaget, 1966; Davydov, 1990; Piaget, 1980). This kind of abstraction establishes many different constructional generalization types, which result in new synthesis between certain rules from which new meaning is obtained (Piaget & Garcia, 1989).

Dubinsky (2000), formed his own perspective of Piaget's reflective abstraction thought in his studies. The author presented his characterization of reflective abstraction as interiorization, coordination, encapsulation, generalization, and reversal. According to his interpretations, interiorization is performed to be able to construct internal processes such as an interpretation of a perceived phenomenon; that is, representation with the help of the ability to use symbols, language, pictures, and mental images. Coordination is necessary for constructing one or two or more new processes. Encapsulation occurs in the construction of new forms, which are related to previous forms, and involves them contextually. When an individual learns to carry out a formed schema in a wider scope of knowledge and experience of phenomena, this schema is considered generalized.

APOS Theory

Dubinsky and McDonald (2001) define mathematical knowledge as the act of interpreting with actions, processes, and objects, and organizing these into schemas to make sense of situations and solve problems. The authors developed the APOS theory to explain the tendencies of an individual and interpret these constructions. According to the theory, action is the transformation of individual instructions (openly or in his/her memory) that are concerned with how objects are perceived by the individual himself/herself. If the action is repeated and its reflection is created by the individual, he/she no longer needs external stimuli. By engaging in actions of similar types, he/she achieves an internal mental construction, called a process. The individual may think about reversing while a process is taking place and may combine this process with others. The object is the construction of the process, in which the individual considers the process as complete and realizes that it can be transformed. A schema for a certain mathematical concept is a background of other schemas, which should be related to a problem case that requires actions, processes, objects, and the mathematical concept. These items are also related to some general principles to constitute a framework in the individual's mind.

Examining the studies conducted on mathematics education (i.e. Asiala, Brown, Kleiman, & Mathews, 1998; Bergsten, 2008; Dubinsky, Weller, Mcdonald, & Brown, 2005; Pinto & Tall, 2002; Stalvey & Vidakovic, 2015; Trigueros & Martinez_Planell, 2010), it is observed that APOS Theory was used for revealing the formation of many mathematical conceptions such as function, infinity, graphic, limit, permutation, and symmetry.

Visualization

Visualization is a complicated process of transforming construction, mental images, and representations. It involves establishing a connection between information about ideas, which are previously unknown, and understandings, which gradually develop. Furthermore, it is a process, ability, and product of the reflection, usage, interpretation, and creation of pictures, images, and diagrams in our minds, on paper, or with technological tools for the purpose of portraying (Bishop, 2003; Hershkowitz, 1989; Wheatley, 1998; Zimmermann & Cunningham, 1991).

Visualization appears not only in the developing of the mathematical thoughts but also the discovery of new relations between mathematical objects, as well as for giving meaning to mathematical concepts and the existing relations between them. It also presents the opportunity to decrease complexity in case a substantial volume of information is encountered. Nevertheless, several educators have pointed out the limitations of and difficulties encountered in visualization, as well as the unwillingness to engage in this task (Arcavi, 2003; Eisenberg, 1994; Stylianou & Silver, 2004). On the other hand, when an individual forms a spatial arrangement, a visual image guides this formation in the individual's mind. For this reason, in doing mathematics, visualization includes all inscriptions of a spatial nature, as well as the construction and transformation processes of visual mental images (Presmeg, 1997). Mathematics pays attention to objectification, representation of abstracting from real situations, and visuality of most of these representations (Bishop, 1989). In this sense, visualization not only organizes knowledge as meaningful constructions but is also a significant factor that guides the analytical development of a solution. According to Zazkis, Dubinsky and Dautermann (1996), visualization is the act of constructing a strong connection between a concept that an individual acquires through internal construction and his/her feelings.

Zazkis et al. have developed a model about visual thinking and called it VA (Visualization-Analysis) model (1996). This model includes terminology about discrete levels of visual and analytical thinking. Even though the levels are discrete, the approach is quite continuous. When the horizontal movement is repeated in the model, each movement in the analysis, which occurs according to the previous movement of visualization, does not only produce an enriched movement but also undergoes a more sophisticated analysis. At first visualization and analysis movements are viewed as distinct and different, however, progressively the two types of thinking become more related and the movement between them occurs more often. As the movement continues, just like a skier who goes for the finish line by slaloming, both thinking styles become more internalized in a person's mind; analysis and visual understandings become synthesized.

Presmeg (1985) regarded visualization as a support for understanding or a tool that enables result generation. She stated that visualization can be spoken of in terms of a concept or a problem, not of a diagram. In visualizing a concept or a problem, the visualization is applied to a mental image of the problem; visualization here means understanding the problem in the diagram or in terms of the visual image. Thus, the visualization process includes visual images with or without a diagram as a necessary part of the solution method. Also she stated that that visual images form in the mind and a visual image is a mental scheme, in which visual or spatial information is conceived. This mental scheme emerges whether a visualized perception exists.

From her observation of students, Presmeg (1986) categorized visual images into five different types: concrete imagery (a holistic picture, in which the parts of everyday objects come together and are constructed in the mind); pattern imagery (simple relationships, which are described in a visual-spatial scheme); memory imagery of formulae (formation of formulae, which we generally see with mental capacity and which are written on a board or notebook in our memory); kinesthetic imagery (images requiring muscle strength activities); and dynamic imagery (active images).

Guzman (2002) argues that the visualization of an individual is not only a process that requires optical processes but also a more superficial phenomenon that we call vision, in which a psychological sense also exists. In this sense, he classified visualization into four types: isomorphic visualization (representations, in which visually manipulated objects can be transformed into mathematical relationships); homeomorphic visualization (subjective representations, in which some elements have certain mutual relationships, and imitating the relationships between abstract objects encourages the construction of images in processes such as guessing, research, and proving); analogical visualization (in which behavior is known and examined beforehand or in which we obtain representations of the behavior more easily by replacing objects that are related to each other analogically and mentally); and diagrammatic visualization (in which mutual relationships and our mental objects are represented with diagrams, facilitating the visualization of our thinking processes).

In light of what is mentioned above, when abstraction process is characterized as interiorization, coordination, encapsulation, generalization, and reversal (Dubinsky, 2000), the visualizations' place in the process becomes convenient for examination. It is possible to determine what visualizations and visual imageries are present in the process and how they are formed through visualization types such as isomorphic, homeomorphic, analogical and diagrammatic visualizations (Guzman, 2002) as well as through visual imagery types such as concrete, pattern, memory imagery of formulae, kinesthetic and dynamic imageries (Presmeg, 1986). It is assumed that APOS Theory (Dubinsky & McDonald, 2001) could be significant in terms of revealing how the knowledge inside one's mind is formed, while carrying out these examinations.

Method

In this study, perception and events were observed under natural conditions in a realistic and holistic manner. A case study was conducted for extensive and in-depth analysis. The essence of a case study is that it tries to illuminate a decision or a set of decisions: why the decisions were made, how they were implemented, and with what result (Schramn, 1971; Yin, 2003). We investigated the role and the importance of visualization on the abstraction process. We determined which visualizations were used by the participants, why and how they established these and what kind of visual images they formed.

In line with this purpose, 24 prospective mathematics teachers, who were senior students taking a course opened at the fall semester, were observed. One of the reasons why we chose to observe students during this course, which is a selective course related to geometry, is because the conception, which is planned to be used throughout the abstraction process in the study, is related to geometry and because the prospective teachers taking this course possess the necessary prerequisites. Thus, it is believed that they have the required background to abstract the conceptions related to the course. The observations were concerned with the prospective teachers' visualizations in the process of abstraction as well as whether they used visualizations throughout the process, how and where they used them, if they used any. So, they are unstructured and visualization-focused observations. This process is also believed to shed light on the selection of participants and the construction of the interviews to be conducted. While selecting the participants, extreme or deviant case sampling was employed since it projects the study of cases, which are supposed to undergo an elaborated examination and is limited in number yet still rich in knowledge (Yıldırım & Şimşek, 2006). For that purpose, the prospective teachers taking the course were briefly interviewed. In these interviews, the prospective teachers were asked unstructured, short, and general questions about their background in mathematics (i.e. basic geometric conceptions) and their transcripts were evaluated. However, grade point averages were not directly effective on the selection of participants, yet they provided information on the general success of the prospective teachers. Therefore, 5 voluntary participants (3 female and 2 male), who were considered capable of completing the process of abstraction, were selected out of all of the prospective teachers. The reason why only those, who were considered capable of completing the process, is because the focus of the study is to reveal the visualizations in the process of abstraction, thus to retrieve rich data about where, how, and how many visualizations they conducted in the whole abstraction process. In light of all of these observations and interviews the participants with different ways of thinking, who used or did not use visualization in the abstraction process were selected according to extreme or deviant case sampling. The case and criteria required to convey the obtained results were made as clear as possible.

Each interview conducted with the participants took one and half an hour on average and they were videotaped. The objective of videotaping the interviews was to elaborately examine the visualizations and visual imageries in the process and to re-watch it at various times with the aim of preventing data loss. Some semi-structured questions were presented to the participants and the interviews were intended to determine how and how frequently the participants used visualization within the mathematical abstraction process. The kind of visual images they included in this process and how they used these images were also determined. They were expected to abstract the statement, “Objects are congruent if they are transformed into each other by four motions: translation, rotation, reflection, and glide-reflection”. To do this, nine levels, which each one administered in three steps, were assigned to each participant. The first step was the congruence of two geometric objects in the real line, the second was the congruence of two geometric objects in the Euclidean plane, and the third was the congruence of two geometric objects in three dimensional Euclidean space. The abstracting levels were about the line segments, the types of geometric shapes and solid geometric objects. In these steps, the researchers tried to create environments, in which the participants could think either by using visualizations or not throughout the process of abstraction, about various objects/shapes. Thereby, it was aimed to see whether they needed visualization, which is the focus of the study. Moreover, by using open-ended questions, it was aimed to get a better view of abstraction processes and to determine the place and type of visualization in these processes. Afterwards, all of the questions determined to be used in the research were separately evaluated by the experts in the field and eventually created by coming to a mutual agreement together. Thereby, nine questions, which applied to different motions of geometric objects/shapes, were created. One of these motions was at real axis; five of them were at Euclidean plane; three of them were at Euclidean space. All of these questions were leveled according to the process of abstraction. Table 1 lists the levels assigned to the participants within the process.

As the participants completed the stages step by step during the interview, the “interiorization, coordination, encapsulation, generalization, and reversal” (Dubinsky, 2000) of the abstraction process were analyzed for each stage in conjunction with data analysis. These analyses were carried out to examine the place and type of visualizations used within the process. In this process, APOS theory (Dubinsky & McDonald, 2001) was employed to explain the mental constructions as these processes were being experienced. During the data analysis, we examined the type of visual images (concrete, pattern, kinesthetic, dynamic, and memory imageries of formulae) (Presmeg, 1986) that occurred in the participants’ minds throughout the process and which type of visualization (isomorphic, homeomorphic, analogical, and diagrammatic visualization) (Guzman, 2002) they implemented.

The data were examined by content analysis, in which we attempted to reveal hidden realities within the data. We grouped similar data within the frame of certain concepts and themes, then arranged and interpreted these (Creswell, 1998; Patton, 1990). First, the data were collected in video form (interviews) and then transcribed. Subsequently, the sentences in the interview transcripts were coded using coding techniques and analyzed comparatively.

In APOS Theory, the genetic decomposition, which is a hypothetical model of how a person’s understanding about mathematical conceptions is built, consists of actions, processes, and objects definitions of the schema that is created through conception building (Stalvey & Vidakovic, 2015). During coding and their linking together with their categories, a genetic decomposition of the thinking process of a participant was first conducted. The following presents a genetic decomposition, which may describe the findings and interpretation of the findings:

a schema of real numbers requiring a number concept as the object and arithmetical algebraic operations as processes; a Euclidean plane schema requiring a point concept as the object and its relationships as processes;

actions related to a real number line concept; actions related to the Euclidean plane; actions related to three-dimensional Euclidean space; and a transformation schema that includes the concepts of motion on the real number line, the Euclidean plane, and the three-dimensional Euclidean space as processes; this schema also requires the coordination of operations related to actions. Recall here that a transformation is called a motion if the transformation is a translation, a rotation, a reflection or a glide-reflection. As the abstraction processes that the participants engaged in were examined, the steps given were studied incrementally and the analysis of each level was carried out as follows.

Table 1. The levels for participants' abstraction processes

1. A(-1), B(3), C(2), D(6) points are given. If you think $[AB]$ and $[CD]$ line segments, can you have one of these from the other with the help of a motion? Why? Can you have it? What do you think about the lengths of these line segments?
2. A(-2,1), B(0,3), C(2,-3), D(4,-1) points are given. If you think $[AB]$ and $[CD]$ line segments, can you have one of these from the other with the help of a motion? Why? Can you have it? What do you think about the lengths of these line segments?
3. O(0,0), A(4,0), B(0,4) points are given. If you think $[OA]$and $[OB]$ line segments, can you have one of these from the other with the help of a motion? Why? Can you have it? What do you think about the lengths of these line segments?
4. If you think OAB triangle with O(0,0), A(2,0), B(2,1) corners and OCD triangle with O(0,0), C(0,2), D(-1,2) corners, can you have one of these from the other with the help of a motion? What do you think about the angles of these triangles? What do you think about the edges of these triangles? What do you think about the areas of these triangles? Do you notice anything?
5. If you think ABCD quadrangle with A(-5,-3), B(-2,-2), C(-1,-1), D(-4,1) corners and EFGH quadrangle with E(0,1), F(3,2), G(4,3), H(1,5) corners, can you have one of these from the other with the help of a motion? What do you think about the angles of these quadrangles? What do you think about the edges of these quadrangles? What do you think about the areas of these quadrangles? Do you notice anything?
6. If you think a circle with A(-3,4) center that have a radius of two units and a circle with B(2,-1) center that have a radius of two units, can you have one of these from the other with the help of a motion? What do you think about the circumferences of these circles? What do you think about the areas of these circles? Do you notice anything?
7. If you think a sphere with A(1,2,3) center that have a radius of one unit and a sphere with B(4,1,2) center that have a radius of one unit, can you have one of these from the other with the help of a motion? What do you think about the areas of these spheres? What do you think about the volumes of these spheres? Do you notice anything?
8. If you think a cylinder with A(2,0,0) center that have a radius of two units and a height of three units and a cylinder with B(-2,0,0),center that have a radius of two units and a height of three units, can you have one of these from the other with the help of a motion? What do you think about the areas of these cylinders? What do you think about the volumes of these cylinders? Do you notice anything?
9. If you think a pyramid with A(-5,-4), B(-3,-1), C(-6,1), D(-8,-2) bottom corners that have a height of four units and a pyramid with E(4,-1), F(6,2), G(3,4), H(1,1) bottom corners that have a height of four units, can you have one of these from the other with the help of a motion? What do you think about the areas of these pyramids? What do you think about the volumes of these pyramids? Do you notice anything?

In the real line: A schema was generated in that two line segments were congruent. This congruence was achieved because a line segment on the number line was transformed into another line segment with motion. Here, the point concept was taken as the object on the number line, the distance between two point concepts and the point concept on the number line were internalized, and then the schema related to the co-linear line segments produced an object that was obtained by thinking about the lengths of the line segment. The object was also obtained by encapsulating the process by which the line segments became congruent as a result of the coordination between the motion concept on the number line and the line segment concept.

In the Euclidean plane: A schema resulted from the congruence of two line segments. This congruence was due to the transformation of a line segment into another line segment. First, the point concept was taken as the object in the plane, and then the line segment concept in the plane and the distance between two point concepts in the plane were internalized. After this, the schema was generated given that the co-linear line segments resulted in an object that was obtained by thinking about the lengths of the line segments. The object was also obtained by encapsulating the process by which the two line segments became congruent as a result of the coordination between a motion in the plane and the line segment concept. Same thoughts were intended for triangle, quadrangle and circle. Encapsulation requires actions such as maintaining angles, edge lengths, radii, circumference, and areas of motion for triangles, quadrangles, and circles.

In the Euclidean space: A schema was produced from the congruence of two spheres. The congruence resulted from the transformation of a sphere into another sphere. First, the point concept was taken as the object in the space, the sphere in the space concept and the distance between two points in the space concept were internalized, and then the schema was obtained because the congruent spheres resulted in an object that was obtained by thinking about the area and volume of the sphere. Encapsulating the process by which the two spheres became congruent also generated the object because of the coordination between a motion in the three-dimensional Euclidean space and the sphere concept. Same thoughts were intended for cylinder and pyramid. Encapsulation requires actions such as maintaining angles, edge lengths, volumes, and areas of motions. As these schemas were formed, transforming geometric objects into congruent geometric objects necessitated the characteristics of congruent geometric figure objects, such as congruent line segments, triangles, quadrangles, and circles. The actions encapsulated in the schemas of plane motions were also required. Furthermore, coordination with the processes that occurred through internalization with actions such as height, volume, and area calculation was also necessary.

Figure 1 displays the organization of the schema, which was created as a result of the motion made by quadrangle as an example. For an extended version for our genetic decomposition, refer to Yilmaz (2011).

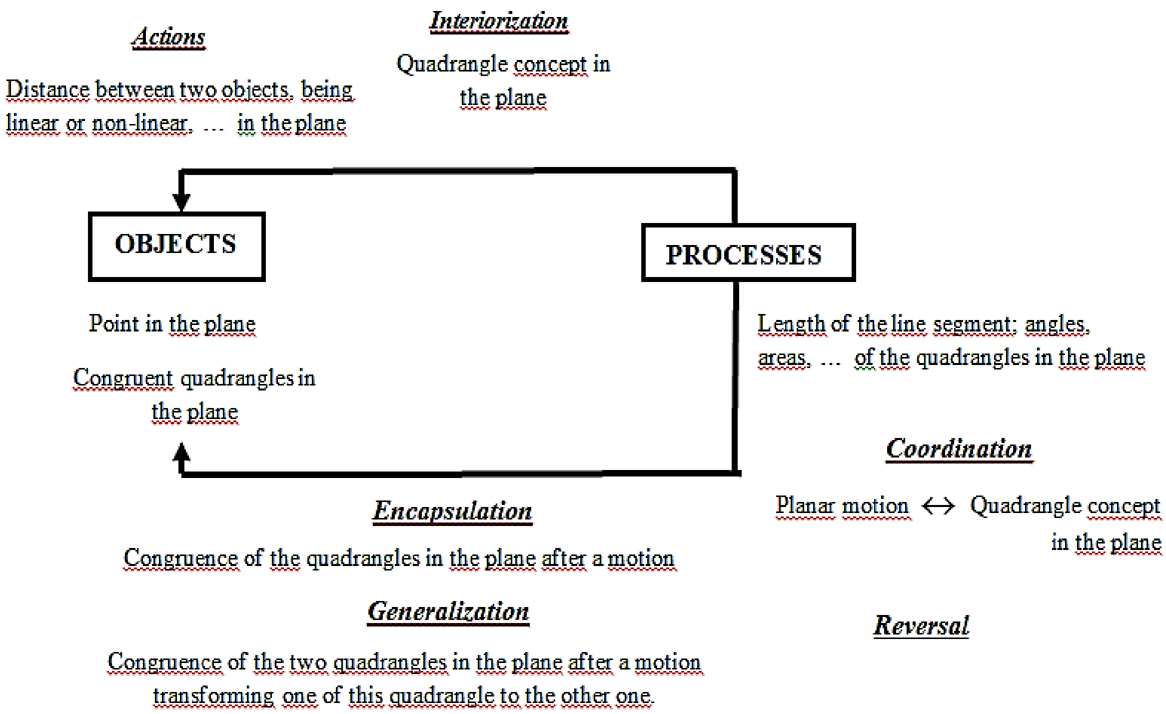

Figure 1. The Organization of the schema created by equilateral quadrangles obtained through the motions on the plane

Findings

In this section, the obtained findings were organized within the framework of what is mentioned above. The situations of the participants, who were given nicknames, were exemplified and described. The abstraction levels were used as base while organizing the findings; the schemas created within the genetic decompositions of the participants were expressed. In this way, if the participants chose to use visualizations, it was aimed to reveal at which levels of abstraction they created visualizations and how and where they created schemas as well as what kind of visual imageries they had. The findings of the study were presented in a clear manner through quoting the participants, which stated the views of them. While exemplifying the quotations of the participants, we tried to include those of who were believed to describe the situation in the best way, even in cases in which a participant is exemplified more than once at different levels. All of this process were reviewed and evaluated with experts from different fields.

Table 2 displays the visualizations the participants used; whereas Table 3 presents the types of visual imageries and their places in the process altogether. As displayed in Table 2, the participants used isomorphic (35 out of 48) and homeomorphic (13 out of 48) visualizations; 17 isomorphic visualizations emerged in the internalization process; 13 of them arose in the coordination process; and 5 of them emerged as the participants handled the beginning object. Meanwhile, 2 homeomorphic visualizations were used in the internalization process; 2 of them were applied in the coordination process; and 9 emerged as the teachers handled the beginning object.

Table 3 suggests that the participants allowed for different visual images at different periods during the abstraction process. Among 142 visual images that emerged, 40 were concrete, 38 were dynamic, 35 were kinesthetic, and 29 were formulae. 112 of them were seen in the coordination process; 21 of them were observed in the internalization process; 5 arose in the reversal process; 3 occurred as beginning objects; and 2 emerged in the encapsulation process. Below presents examples of the quotations made by the participants for each level. Moreover, coding samples in the process of analysis are demonstrated in *italic* form.

In the first level, for instance, during Gonca's visualization, she regarded the points that she expressed as concrete imagery in the *plane* instead of considering the real axes (Figure 2). As she tried to internalize the line segment concept, while thinking of *the length of the line segment* she kept track of the line segments using her fingers an expression of her kinesthetic imagery by saying, *"If I look at their length... the length of the AB line segment and the length of the CD line segment are 4 units… equal"* (Figure 3). She also used her dynamic imagery by saying, *"That is at the rate of a certain unit in order to bring point B to point C."* Using her formulae image for the line segments, she stated, *"Let me compare the coordination of the point".* She looked at the *components* of the line segments by using formula imageries and realized they had *the same length.*

While trying to find coordination between the motion concept and line segment, she realized that a motion is *a translation,* thus expressed it as *proceeding*. As she thought about the figure she visualized homeomorphically, she made a mistake in translating the points. Also, as she thought about the translation during reversal, she said, *"Let's take the x and y coordinates in plane…"* By saying this, she visualized her thoughts isomorphically, which helped her understand that the motion occurred on the number line. She saw the *amount of translation* on the figure and *made them cross* by *bringing together* line segments.

In the second level, for instance, Mert regarded the points given as the beginning object and he initially visualized these homeomorphically as he formed his concrete imagery of the line segments. Then, he visualized the points in plane isomorphically and internalized the line segment concept in plane (Figure 4). He said, *"AB and CD have the same length,"* which he determined by identifying the edge lengths of the right triangle on the figure that he drew. As he thought about the transformation of the line segments into other line segments, he pushed these forward with his hand, indicating that he used his kinesthetic imagery. He applied his dynamic imagery when he said, *"Now I have to look at whether I can obtain CD from AB. First, I have considered their slopes. That is, I will think about whether I can dislocate the slopes."* He first thought that the line segments were *parallel* because they had the same slope, and he determined that he could obtain the line segments with *the reflection motion* of these segments, which he expressed as *symmetric*.

Table 2. The Types of Visualizations the Participants used in Abstraction (H: Homeorphic Visualization, I: Isomorphic Visualization)

	OYA		MERT		TANER		GONCA		FATMA	
Level 1	Beginning object Coordination	H I	Beginning object Interiorization	H I	Beginning object	I	Beginning object Interiorization	H I	Coordination	I
Level 2	Beginning object	H	Beginning object Interiorization	H I	Beginning object	I	Interiorization	I	Beginning object	I
Level 3	Beginning object Interiorization	H I	Beginning object	I	Interiorization	I	Interiorization	I	Beginning object	I
Level 4	Beginning object Coordination	H I	Coordination	I	Interiorization	I	Interiorization	I	Coordination	I
Level 5	Beginning object Coordination	H I	Interiorization	I	Coordination	I	Interiorization	I	Interiorization	I
Level 6	Interiorization	I	Interiorization	I	-	-	Interiorization	I	Interiorization	I
Level 7	Interiorization	H	Coordination	I	-	-	Coordination	I	Interiorization	H
Level 8	Beginning object Interiorization	H I	Coordination	I	-	-	Coordination	I	Coordination	H
Level 9	-	-	Coordination	I	-	-	Coordination	H	Coordination	I

Table 3. The Visual Imageries of the Participants in Abstraction (Concrete imagery: C, Dynamic imagery: D, Formulae imagery: F, Kinesthetic imagery: K)

	OYA		MERT		TANER		GONCA		FATMA	
Level 1	Interiorization Coordination	D,K C,D	Interiorization Coordination Encapsulation	C D,K K	Coordination	C,D,K,F	Beginning object Interiorization Coordination	C D,F,K D	Coordination Reversal	C,D F,K
Level 2	Interiorization Coordination	C F,K	Beginning object Interiorization Coordination	C D F,K	Interiorization	C,F,K	Interiorization Coordination	C,F D,K	Interiorization Coordination Reversal	C D,K F
Level 3	Interiorization Coordination	C,K D,F	Interiorization Coordination	C F,K	Coordination	C	Interiorization Coordination	C,K F,K	Coordination	C,D,F,K
Level 4	Beginning object Coordination	C C,D,F,K	Coordination Reversal	C F,K	Coordination	C,D,K	Interiorization Coordination	C F,K	Coordination	C,D,K
Level 5	Coordination	C,D,F,K	Interiorization Coordination	C,F D,K	Coordination	C,D	Coordination	C,D,F,K	Coordination	C,D,K
Level 6	Coordination	C,D,K	Coordination	C,D,K	Coordination	D,F	Coordination	C,D,F,K	Coordination	C
Level 7	Coordination	C,D	Coordination	C,D,F,K	Coordination	D,F	Coordination	C,D,F,K	Coordination	C,D,F
Level 8	Coordination	C,D,K	Coordination	C,D,K	Coordination	D,F	Coordination	C,D,F,K	Coordination	C,D
Level 9	Coordination	F	Coordination	C,D	Coordination	D,F	Coordination	C,D,F,K	Coordination	C,D,K

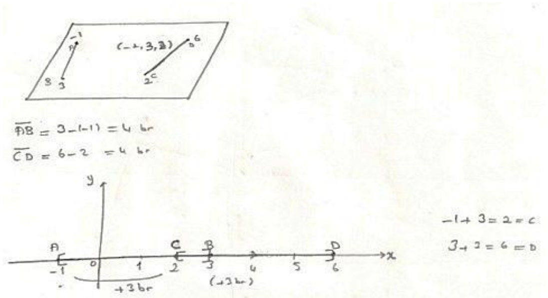

Figure 2. Representations of Gonca in level 1

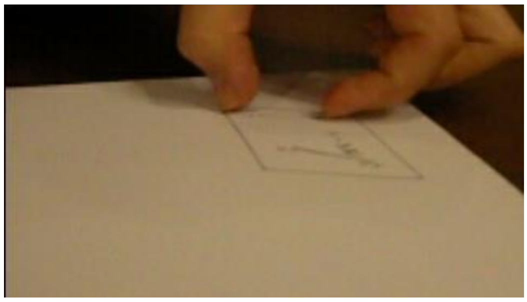

Figure 3. Picture of Gonca in level 1

However, he thought that the necessary values as his formulae imagery in his mind could not be obtained and decided that the motion was the *translation*. Meanwhile, he thought about the distance between points and obtained a translation vector. As he coordinated the motion concept and line segment concept, he used the expression of *vectors* for these line segments, after which he obtained the encapsulation of the process by which they became *congruent*.

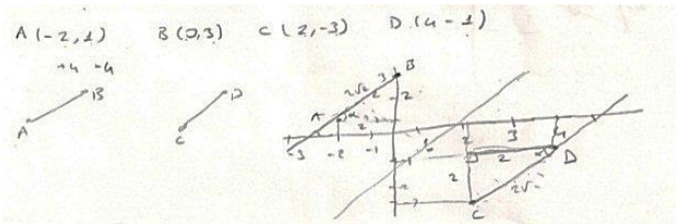

Figure 4. Representations of Mert in level 2

In the third level, at first, Oya homeomorphically visualized the points she considered the beginning object and the line segments that these points formed. Then, she thought that the line segment could not be transformed into another, according to the first figure that she visualized (Figure 5). She expressed this as, *"I cannot obtain it here... its point of origin hasn't changed…"* Her visualization caused her to think incorrectly. Afterwards, she internalized the act of thinking about the distance between two points on the coordinate plane and line segment concept. She visualized the line segments isomorphically. She said, *"Now, here, a 90-degree rotation was applied,"* and as her kinesthetic imagery, she applied the *rotation* by holding the first line segment with her fingers and bringing it onto the second line segment. As she coordinated the plane motion concept and the line segment concept, she visualized isomorphically and obtained the *equation for the rotating motion* with the help of her formulae imagery. At the end of this process, she obtained a line segment, which she retrieved from another line segment, and used dynamic imagery by saying *"Only its place changes, that's how I perceived it"*.

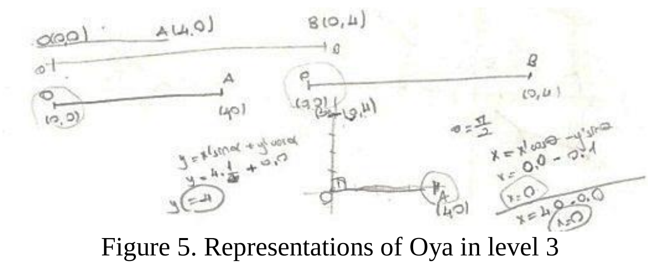

Figure 5. Representations of Oya in level 3

In the fourth level as her concrete image, Oya homeomorphically visualized the points that she considered the beginning object. Then, she internalized the act of thinking about the distance between the two points in the plane and the triangle concept in the plane. As she coordinated the two-dimensional motion concept and the triangle concept, she visualized the triangles isomorphically by saying; *"Now, let's draw the coordinate system"* (Figure 6). To specify the two-dimensional motions, she moved the first triangle (as her concrete imagery) on the figure that she drew to the second triangle (as her kinesthetic imagery) with her pencil and had, 90-degree *rotation.* Meanwhile, she expressed her dynamic image thus: *"….if we take it here …and we take it to the top…it goes here….."* As she explained this action, she articulated that the points *coincided* by obtaining the vertexes of the point that was rotated from the vertex points in the equation for the rotating motion. She presented this equation with her formulae imagery. In conclusion, the angles, sides, and areas of moving triangles *did not change*, yet, *their places changed.*

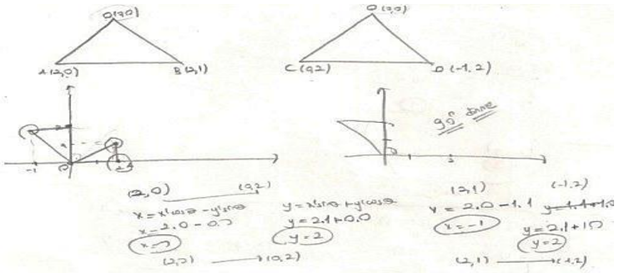

Figure 6. Representations of Oya in level 4

In the fifth level, Oya regarded the point concept in the plane as the beginning object and internalized the line segment and quadrangle concepts as she thought about the distance between two points by homeomorphically visualization. As she coordinated the two-dimensional motion concept and the quadrangle concept, she started to visualize the quadrangles, which she isomorphically expressed as her concrete imagery by saying*, "If we think about the coordinate plane…"* (Figure 7). She thought, "*Let me find the distances and then do a matching according to these…"* however, she applied isomorphic visualization because she thought this would be insufficient for understanding how the motion occurred. Then, she calculated the edge lengths of the quadrangles with her formulae imagery, visualizing in a homeomorphically manner. She moved the quadrangle's edges with her fingers by rotating the figure she visualized isomorphically as her kinesthetic imagery, and she used her dynamic imagery by saying, *"…all in all, this is the dislocation of a certain edge".* Next, she thought of the *amount of translation* and saw that it was proved. She stated that since *the components of point*, which builds the edges of the quadrangles, will increase proportionally, the angles and areas of edges *will not change* and *remain equal.*

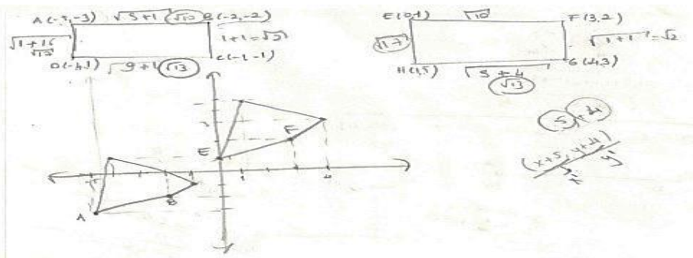

Figure 7. Representations of Oya in level 5

In the sixth level, Oya stated that the motion would be the *translation*, according to the figure that she expressed as her concrete imagery during the coordination of the plane motion concept and circle concept (Figure 8). In this process, she indicated her dynamic imagery by saying, *"If we take that down by 1 unit, that is, we do translation, it goes 1 unit down. Then its center moves there but something like this happened… next, if we rotate…this time, it moves here. That is, we obtained a circle, like this. If we bring the circle to this direction by 1 unit, I mean, the translation, we obtain the same circle. All in all, I think the circles coincide when we rotate the circle like this."* As her kinesthetic imagery, she moved the circles with her fingers (Figure 9). She did so by counting it on the axis unit by unit. Thereby, she saw how translation occurs and obtained *translation vector.* She stated that the radius of these two circles were the same, thus their areas and circumferences were *equal*, and also *all of their characteristics would be the same.*

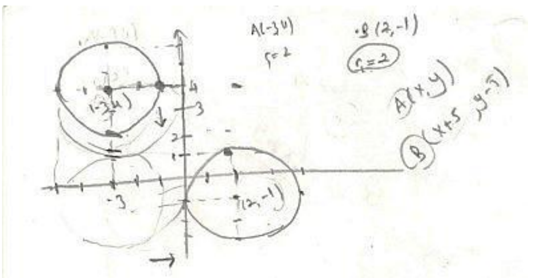

Figure 8. Picture of Oya in level 6

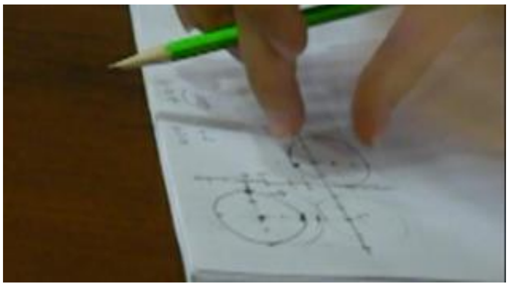

Figure 9. Representations of Oya in level 6

In the seventh level, for example, during the coordination of the three-dimensional motion and sphere concept, Oya could not express the spheres as her concrete imagery, causing her difficulty in expressing her opinion about how the motion would occur. She tried to visualize the motion isomorphically but failed to do so. She looked at the differences between the components of the centers of the spheres by saying, "*The lengths of the radii are equal,* and she interpreted this as the motion of *translation"* (Figure 10). She expressed her hesitation by saying, *"In fact, as I couldn't draw these well, that is, they would probably be the translation but how can I explain…? I mean… moving it to the three-dimensional space perplexed me. I wish I can understand whether I can draw accurately. …If I had something in hand…"* and added, *"I think their areas and volumes are equal as their radii are equal, too. However, as I couldn't see, I cannot be sure."* In order to explain the motion, she tried to draw the figure for the second time and said *"I mean, over the axis… I think points should be moved according to axes",* thus used dynamic imagery. In conclusion, she made a statement, which showed that she was not sure of her views: "*Since their radius is equal, I believe their areas are also equal. When we consider their volumes, they are equal, too. In other words, everything they have is equal. However, I cannot be sure since I cannot see it… It only feels like they would be equal…"*

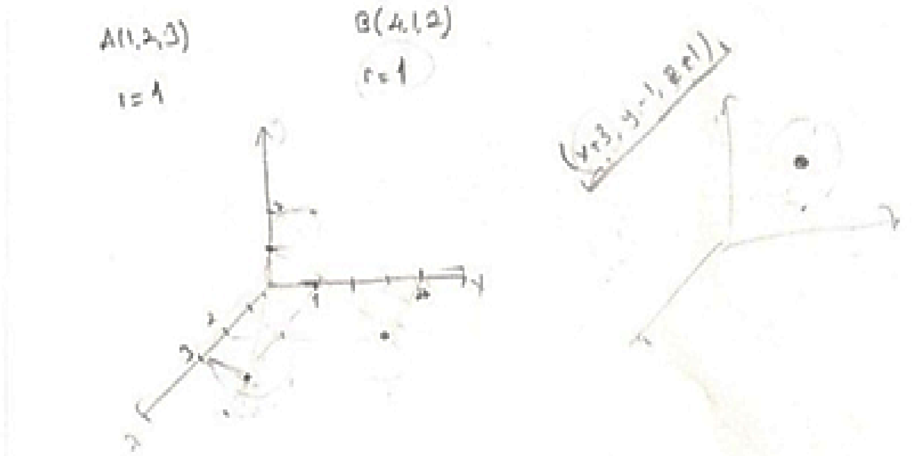

Figure 10. Representations of Oya in level 7

In the eighth level, Mert initially considered isomorphically visualizing the cylinder segments given on paper, and then he gave up (Figure 11). During the coordination between the motion concept in three-dimensional space and cylinder segment concept, he modeled the pencils as though these were three-dimensional base vectors and explained his opinions with a concrete cylinder segment (Figure 12). He said, *"The cylinder is here when it is at -2. Its radius was here. When its center was (2,0,0), it was here. When its center was (2,0,0), I put the center to (-2,0,0). That is, I did a translation at only one dimension."* He showed his dynamic imagery by saying, *"I obtained this through translation. Nothing changed."*

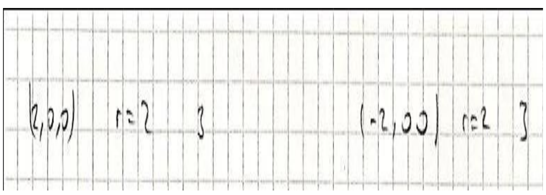

Figure 11. Representations of Mert in level 8

Figure 12. Picture of Mert in level 8

In the ninth level, Taner used his dynamic imagery by saying, *"If I translate each of these base edges by (9,3), the bases of these two pyramids coincide with squares and quadrangles, which frame their bases. Therefore, these two pyramids coincide since their heights are the same. That is, these can be equalized through translation."* For the radix edge points of the pyramids, he obtained a translation vector and the act of thinking about the distance between the two point concepts, which he expressed as his formulae imagery (Figure 13).

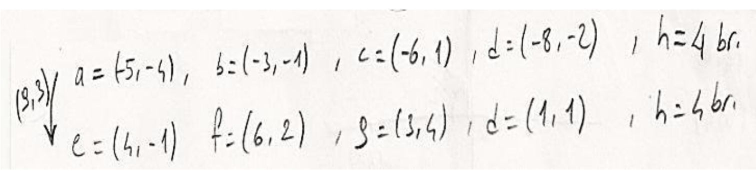

Figure 13. Representations of Taner in level 9

During the coordination between the three-dimensional motion concept and pyramid concept, Gonca initially did not want to use visualization. She eventually decided to apply homeomorphic visualization and explained this with, *"When I drew the most efficacious object, I made an error. In fact, I see better when I draw a figure. But it became a bit strange not to be able to draw the correspondence between the most efficacious objects in my mind. I tried to write them down as they are three-dimensional."* She continued, saying, *"If I indulge in it without drawing correctly, like that, we will be better off but…. If I draw the coordinate, it will be better but…Or let me make it here, at least I see what it is…"* Gonca had difficulty expressing the pictures in her mind as her concrete imagery because she could not visualize the correspondence of her opinions even as she correctly drew on paper. She thought that the motion was the translation, according to the figure she drew (Figure 14) and added, *"…it can be seen easily according to the figure."* She looked at the distance between the base edge points of the pyramids and tried to compare these to obtain the translation vector with her formulae imagery. She used her dynamic imagery by saying, *"I think that point A will coincide with point E …"* She also said, *"Point AB changed into this. Similarly, as their heights are the same, these lengths are equal when we think about the top points here…of these triangles… Again, if we think in sections, congruent triangles are obtained because of translation. So these areas are saved here since the areas of the triangles are saved in translation. Hence, the complete area is saved.* She demonstrated the motion of the pyramids by moving the objects with her hands, this being her kinesthetic imagery.

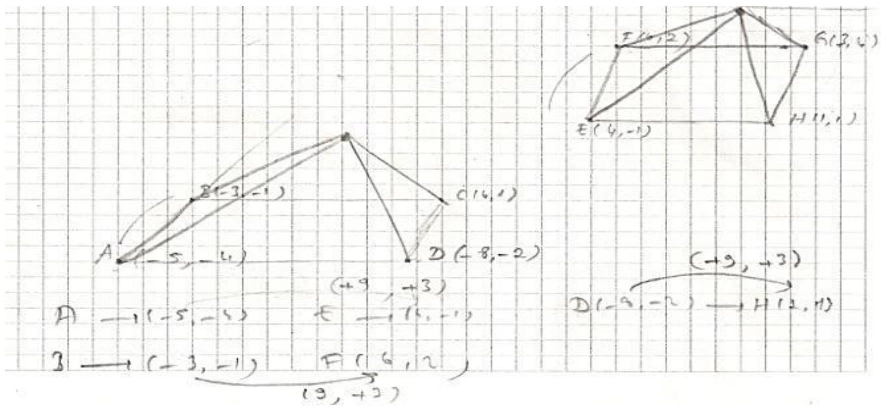

Figure 14. Representations of Gonca in level 9

Discussion and Conclusion

In this study, pre-service secondary mathematics teachers applied different visualizations in different steps of the abstraction processes, and although they used these constantly, they did not actually need to. Presmeg (1986) defines the mathematical visuality of a person as the preference for using visual methods while he/she is examining mathematical problems that can be solved with the help of visual and non-visual methods. The participants in this study can be included in the visualization group because they generally attached importance to the visualization within the abstraction process. Also, Piaget (1980) stated that abstraction has been referred to as the general coordination of actions, with the source of abstraction being the subject, which is realized internally. Constant exercising of visualization during coordination and internalization has shown that visualization has a significant role in the abstraction process.

The pre-service teachers participated in this process sometimes by using visualizations and sometimes with the help of algebraic operations. They needed to support the visualizations they used with algebraic operations. In addition, they tried to interpret the results they obtained algebraically with the help of visualization. The relationship and motion between these two thinking styles occurred within the process quite closely. Once the participants obtained access to a concept (related to a particular stage), which was abstracted within every level to get closer to access, these two thinking styles synthesized themselves. In that sense, this result substantiates the VA model of Zazkis et al. (1996). The participants also sometimes felt the necessity to explain or support their opinions, which they specified algebraically within the process via visualization. At other times, they sought the help of visualization in formally expressing the cases that they experienced intuitionally. Furthermore, they looked for a benefit from visualizations as they handled and recovered the conceptual supports, which can be omitted easily with formal analysis. The results revealed that the visualizations that the participants engaged in are important in supporting the concepts and the relationships between them. These results are also consistent with Arcavi's idea (2003) of visualization having a strong complementary role.

Eisenberg and Dreyfus (1991) stated that some students dislike visualization and prefer using algorithms, although there is a general tendency toward visualization. The authors also emphasized that one of the most basic reasons for this reluctance is the high cognitive requirement in visual thinking. In the current study, some of the participants experienced difficulty in applying visualization even though they needed it within the process. For this reason, the participants tried to explain their ideas only formally. This situation proves the existence of the tendency to not choose visualization and the reasons behind it, as discussed by the researchers. Furthermore, this situation prevented the participants from forming relationships and sometimes caused them to create contradictory ones. Hence, it is possible to say that the reason behind not choosing visualization prevents people to make associations.

One of the important results that emerged within the abstraction process was the inadequacy of the participants in the use of algebraic operations and formulae. This result coincides with the emphasis of Hershkowitz, Schwarz, and Dreyfus (2001) on the abstraction process as being "based on personal background of the individual who struggles with the solution." The participants tried to complete the lacking parts by using visualizations so as to establish relationships with concepts within the development of the process.

Visualization within the abstraction process took much more time than did non-visual methods. Conversely, some of the participants who used visualization could not remember some key points that were necessary for analytical solutions. The participants who visualized incorrectly applied inappropriate links, a situation that negatively affected the process. Such difficulties that emerge from visualization can be thought of as a disadvantage.

The study demonstrated that the participants had different visual imageries. Dynamic images were the most commonly used in the coordination process and concrete images were used considerably more frequently in the internalization process. Linking the visualizations applied showed that among the participants, the ones who used more visual images applied visualization more frequently.

Concrete images generally emerged as scenes that represent pictures related to concepts in the minds of the participants. These mental scenes were sometimes seen when the participants dealt with the beginning object, sometimes in the internalization of the concept, and mostly during the general coordination of the motions in the construction. Concrete images accounted for a significant portion of the visualizations of the participants, and the ones who had a clear concrete image related to the concept accomplished visualization more successfully. By contrast, the desire of the participants who did not have a clear concrete image to carry out the process in a visualization-assisted manner confronted them with difficulty in expressing their ideas. When they visualized the incorrect concrete image, the participants evaluated the process incorrectly. The dynamic images used were similar to the mental movement of the related concepts. To link a concept with different concepts, these movements were implemented mentally, as changing places. These images, which commonly emerged during the coordination process, were sometimes observed when the concept was internalized. The kinesthetic images formed as the participants showed the objects related to the concept in air and on paper, as well as with their finger, hand, or arm movements. These demonstrations were realized largely during coordination.

The formulae images, which the participants visualized in their mind, emerged primarily during the linking of the motions related to the concepts within the coordination process. They were used in the internalization process; sometimes they occurred during reversals. Given that most of the participants felt the necessity to support their ideas algebraically even if they used visualization within the process, this situation stimulated uncertainty in expressing opinions for the participants who were lacking in formulae images. By contrast, the participants who had a clear formulae image carried on with the process more confidently.

In the present research, the participants who generally formed concrete and dynamic images and applied these largely within the coordination process emphasized the role and importance of visual images within the abstraction process-a result that coincides with the idea of Presmeg (1986) about concrete images being effective in the alternation between non-visual cases in the abstract mode. The result is also consistent with the author's assertion that dynamic images are potentially effective visualization tools.

Recommendations

In mathematics education, consciously establishing concepts and relationships necessitates the organized construction of an abstraction process. As demonstrated in this research, visualization plays an important role in the development of abstraction. Hence, providing students opportunities to develop their visual abilities and encouraging them to think visually are appropriate measures. If the students are encouraged to create pictorial portraits related to concepts, they can easily determine whether they have created a visual image related to the concepts, and evaluate the accuracy or inaccuracy of this image (if one is formed). In this way, the accurate construction of a concept is enabled. In steps wherein individuals do not have a clear image of the concept abstracted or formed an incorrect image, the abstraction process can be supported with appropriate visuals. Furthermore, teacher can motivate students to form symbolic expressions of images through sensitive transformation from the visual to the symbolic.

Given the important difficulties that students encounter in coordinating concepts, the use of appropriate visualizations will be especially beneficial in filling the gaps in the processes. Moreover, because concrete and dynamic images are the most commonly used images, visual support for these images will bring forth positive effects on the development of the abstraction process.

Sometimes visualization is considered as unnecessary and even avoided because it is assumed to take too much time and to usually require high cognition as well as due to the belief that real mathematics cannot be done through visualization. That’s why, primarily teachers should be conscious of the subject of visual thinking and prospective teachers should be raised in accordance with that objective. For that reason, it is believed that it could be beneficial to conduct and evaluate more studies on visualization in the institutions, which raise prospective teachers. This study has focused only on visualizations and visual imageries on the abstraction of one concept and has been conducted with a limited number of prospective teachers. It is believed that carrying out researches, which study visualization processes of different conceptions and visualizations of different processes such as problem solving and modeling, and the studies, whose participants are not only students but also teachers, will significantly contribute to the field.

Note

This study is a part of the first author’s doctoral dissertation.

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Author Information

Rezan Yilmaz
Ondokuz Mayıs University
Faculty of Education
Department of Mathematics and Science Education
Samsun, Turkey
Contact e-mail: *rezzany@omu.edu.tr*

Ziya Argun
Gazi University
Faculty of Education
Department of Mathematics and Science Education
Ankara, Turkey