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Abstract

This study explores the identification of discourse communities within mathematics education using factor analysis. Given the exponential growth in mathematics education research, understanding the evolving dialogues within the field has become increasingly challenging. This study employs a systematic literature review, examining the landscape of research on fractions, decimals, and rational numbers (FDR) over the past four decades. By analyzing articles from 11 top-tier journals, the study identifies distinct discourse communities that have emerged around the FDR topics. The research employs computational techniques, including lexical analysis and exploratory factor analysis, to detect patterns and clusters of terms within the literature. These clusters represent latent variables, or underlying discourse communities, that share common concepts and methodologies. The findings reveal three primary communities: procedural versus conceptual knowledge, social constructivism, and radical constructivism. The study further explores how these communities have shaped the understanding of FDR topics, providing insights into the diverse theoretical frameworks that drive research in mathematics education. By integrating both qualitative and quantitative methods, this study enhances the transparency and reproducibility of the analysis, offering a novel approach to understanding the complex structures within mathematics education research. The results underscore the potential for computational tools to assist researchers in navigating and interpreting the growing body of literature in the field.

Introduction

Over the past decade, research in mathematics education has experienced remarkable growth, particularly concerning teacher and student learning. The field boasts numerous journals, each releasing at least one volume quarterly, resulting in an exponential increase in information flow. However, staying abreast of this wealth of knowledge poses challenges due to constraints in time and resources. Merely skimming abstracts is insufficient for a comprehensive understanding, as thorough and critical reading of research literature is necessary. Consequently, gaining a holistic view of field trends becomes challenging for individuals. To address this obstacle, we utilized modern computational capabilities to conduct a systematic literature review spanning the last four decades. This approach enabled us to sift through a vast array of published resources, examining how mathematics education researchers have explored fractions, decimals, or rational numbers (FDRs) within their work. This review focuses on the contemporary discourse practices within mathematics education, particularly within the realm of rational numbers. Our goal is to elucidate the current landscape of rational number research and to identify distinct discourse communities within mathematics education that are actively involved in this area of inquiry. Our primary research question guiding this endeavor was: What distinguishable discourse communities within mathematics education have investigated fractions, decimals, or rational numbers over the past four decades?

Utilizing a computerized data analysis technique, we have examined lexical trends to offer a visual representation of our findings, employing a metaphor of mitosis to depict subclusters of discourse communities prevalent in research on FDRs. Recognizing the pivotal role of well-executed research reviews in providing practitioners and fellow researchers with insights into current trends in the field, we acknowledge that determining the appropriate time frame for article inclusion and establishing a coherent structure to connect articles across this period are critical tasks. However, this process is inherently subjective and subject to the reviewer's perspective, potentially amplifying some viewpoints while diminishing others. To mitigate potential biases and ensure a balanced and comprehensive review, we have complemented traditional qualitative review practices with quantitative methods. Integrating quantitative methods with traditional qualitative review practices enhances the objectivity and scope of research reviews. By employing statistical analysis and computational tools, this approach systematically analyzes large volumes of text, ensuring comprehensive coverage. Consequently, the review achieves a balanced perspective by corroborating qualitative insights with empirical data, making the findings more robust and credible.

In crafting our methodology, we drew inspiration from Nelson’s (2020) computational grounded theory, tailored to fit the specifics of our data and research question. This approach integrates human expertise in interpretation with the computational capabilities of computers, enabling researchers to employ both close and distant reading methods to deepen their understanding of textual meaning. The methodology involves three distinct steps: i. utilizing computational techniques to extract clear lists or networks of words from complex texts, allowing for the initial detection of patterns; ii. engaging in thorough reading of the data to develop a broader comprehension of the implicit meanings within the textual information; and iii. applying computational methods to validate the identified patterns. This structured approach not only incorporates expert substantive knowledge to guide the formulation of questions and hypotheses about the text but also enhances the transparency, reproducibility, and efficiency of the content analysis process (Nelson, 2020, p. 9). The study was designed to within two stages, as described in this section.

Methods

We narrowed our focus to the 11 "top tier" research journals in the field according to Surveys of Scholars, as outlined by Nivens and Otten (2017), spanning articles published from 1977 to 2018. We chose the timeframe to encompass a comprehensive span of literature up until the start of our project in 2018. This period allows us to capture significant developments and trends in the field over four decades, providing a robust historical perspective on the evolution of research in mathematics education. We acknowledge that further literature has emerged post-2018, and we consider the exclusion of this recent literature as a limitation of our study. Future updates to this review could extend the analysis to include newer publications, thus maintaining the relevance and comprehensiveness of our findings.

Despite this narrower timeframe scope, the sheer volume of relevant articles made it impractical to read them all manually. As a result, we devised an alternative approach involving the application of specific computational analysis algorithms along with a thorough reading of a few articles. By harnessing technology to augment human sense-making processes, we transformed the data into comprehensible elements by identifying clusters of terms. These clusters served as markers to pinpoint specific discourse communities prominent within the field.

Stage 1

Criteria for Selecting Articles

To compile articles discussing fractions, decimals, and rational numbers (FDR) from January 1977 to June 2018, we focused on the top 11 journals, excluding the International Journal of Mathematical Education in Science and Technology, as identified by Nivens and Otten (2017) (see Table 1).

Table 1. Journals and Number of Articles in Stage 1

Tier	Journal name	Year founded	Selected articles
1	Journal of Research in Mathematics Education (JRME)	1977	58
1	Educational Studies in Mathematics (ESM)	1977	67
1	The Journal of Mathematical Behavior (JMB)	1980	53
1	ZDM-The International Journal on Mathematics Education (ZDM)	1997	22
1	Journal of Mathematics Teacher Education (JMTE)	1988	21
1	Mathematical Thinking and Learning (MTL)	1999	18
1	For the Learning of Mathematics (FLM)	1981	28
2	Research in Mathematics Education (RME)	1999	11
2	Mathematics Education Research Journal (MERJ)	1989	39
2	International Journal of Science and Mathematics Education (IJSME)	2003	20

We initially focused on the top 10 journals to ensure a manageable scope and high relevance of articles for our analysis, which led us to exclude the International Journal of Mathematical Education in Science and Technology. However, after further consideration and the exclusion of 'For the Learning of Mathematics' from our selection (discussed ahead), we expanded our pool to include this journal. Utilizing Google Scholar and EBSCO, a team of six researchers conducted searches within each journal using the search terms: “fraction*” OR “decimal*” OR “rational number”. For the Journal of Mathematical Behavior (JMB), our review was limited to articles published between 1994 and 2018 due to the availability of online resources. We only had access to the digital versions of the articles starting from 1994, and, due to resource constraints, we were unable to include hard copy publications in our analysis.

Once articles were preliminarily selected, data from each article were manually recorded in a Microsoft Excel spreadsheet. This included journal name, year, volume, title, author, abstract, keywords, sample, examples of students' work, discussion of students' learning, and theoretical framing. Abstracts were then screened for inclusion criteria based on a close reading of whether the articles explicitly focused on teaching and learning content related to FDR numbers. This screening process resulted in the removal of 14 articles which were identified as duplicates within our collected pool of articles. These duplicates had been inadvertently included from multiple sources, and their removal was necessary to ensure the integrity and uniqueness of our dataset, resulting in a final corpus of 337 unique articles.

Criteria for Generating the List of Terms

After compiling a corpus of 337 articles, we identified the 20 most cited articles. Due to the unavailability of Digital Object Identifier (DOI) information for all articles, automating this process using tools like Web of Science was not feasible. Consequently, we converted the reference sections of each article into text files and standardized these citation records using OpenRefine, resulting in a count of the most frequently cited articles. All 20 of the most cited articles identified were already included within our initial corpus of 337 articles, ensuring that our analysis of key terms and ideas was grounded in the most influential sources documented in our study's dataset. The first and third authors then meticulously read these 20 articles to identify terms commonly used to convey key ideas within the published materials. For example, terms such as "second-order model," "radical constructivism," "reorganization hypothesis," or "abstraction" were frequently utilized by mathematics education researchers employing teaching-experiment research methodologies. This process yielded a list of 172 terms. In our methodology, coding served as a dynamic tool to refine and expand our term list by revealing how certain terms were utilized across different contexts within the articles. By coding a new set of 20 randomly selected articles, we were able to observe additional specialized 40 terms that were not initially apparent. This process allowed us to identify terms that were significant within certain discourse communities but were used in varied ways across different studies. For example, new terms related to specific educational techniques or theoretical concepts emerged during this secondary coding, helping us to capture a broader spectrum of the language and methodologies employed in mathematics education research.

These 212 terms were then deliberated upon by the research team to evaluate their utility in distinguishing communities within the field. Some terms, while specialized within one community, were also used by other communities according to standard definitions. For instance, terms related to representations in teaching fractions, decimals, or rational numbers initially included specific methodologies such as "paper-folding," "base-ten blocks," and "computer-based tools/virtual manipulatives." However, it was realized that coding for such terminologies might be cumbersome and not necessarily provide distinctive features to differentiate between different communities. Consequently, terms were categorized as separating and non-separating. Terms such as "curriculum analysis," "textbook analysis," "pre-service teacher," "in-service teacher," "representations," and "grade level" were deemed non-separating, as they denoted common interests cutting across discourse communities within mathematics education research. The distinction between articles using these terms and those that did not was considered irrelevant to the research interests. Conversely, terms primarily focused on differentiating between discourse communities, such as "misconception," "mistake," "taxonomy," "perturbation," "constructivist," "memorization," or "recursive partitioning," were labeled as separating terms. Despite syntactic variations for the same term (e.g., plural and singular forms of a noun), consensus was reached by discarding some non-separating terms, resulting in the retention of 98 terms.

Given the taxing nature of manually coding the remaining 297 articles using the identified 98 terms and considering the subjectivity of our inclusion/exclusion criteria for the initial group of articles, we opted to streamline and automate the article inclusion process to broaden the scope of the study.

Stage 2

Considering insights gained from Stage 1 of the study, we made the decision to exclude the journal For the Learning of Mathematics (FLM) from the list of top journals and proceeded to automate the analysis process with the remaining 10 journals. Our rationale for excluding FLM stemmed from our observation that papers published in FLM tended to focus more on teaching and discussion rather than on research. In this stage, we revised our criteria for selecting articles by broadening the scope of article selection based solely on the syntactic occurrence of our search terms: fractions, decimals, and rational numbers. This adjustment aimed to minimize subjectivity in determining inclusion criteria. We used the same search string to locate published resources from the selected journals was “fraction*” OR “decimal*” OR “rational number*”. We conducted this search using the Web of Science, covering the period from 1977 to June 2018. This stage of the search yielded 2166 articles (see Table 2).

Table 2. Journals and Number of Articles in Stage 2

Tier	Journal name	Selected Articles
1	Journal of Research in Mathematics Education (JRME)	288
1	Educational Studies in Mathematics (ESM)	470
1	The Journal of Mathematical Behavior (JMB)	252
1	ZDM-The International Journal on Mathematics Education (ZDM)	288
1	Journal of Mathematics Teacher Education (JMTE)	149
1	Mathematical Thinking and Learning (MTL)	51
2	Research in Mathematics Education (RME)	77
2	Mathematics Education Research Journal (MERJ)	155
2	International Journal of Science and Mathematics Education (IJSME)	120
2	International Journal of Mathematical Education in Science and Technology (IJMEST)	316

Criteria for Generating the List of Terms

After acquiring the articles, we omitted the reference sections of each paper to concentrate solely on the textual content. While automating the search criteria facilitated the capture of relevant articles, it also introduced articles into our dataset that necessitated careful consideration. For instance, our use of the key term "fraction" in the search meant that articles containing this term could vary significantly in relevance. Some articles might mention "fraction" only once, and in a context unrelated to the teaching and learning of mathematics, such as in the phrase "only a fraction of the population agreed." To address this variability, we acknowledged that authors deliberately focused on mathematics would likely use the term "fraction" more frequently than those writing about other topics. As a result, we established a threshold for the minimum number of occurrences of a term within an article to determine our inclusion/exclusion criteria. This threshold helped us refine our selection process, ensuring that articles included in our analysis were more closely aligned with our research focus.

Thresholding of Terms

To operationalize our subjective judgment of "aboutness," we focused on the frequency of key terms (fraction, decimal, rational number) in the primary phase of analysis, building on insights from our exploratory phase reading of abstracts. We determined the threshold through qualitative judgment while randomly sampling 40 articles. To systematically screen these articles, we employed the lexical search feature of MAXQDA (version 18.1.1; VERBI Software, 2016), a qualitative data analysis software program. This tool provided information on how many times specific terms appeared within the text of an article, serving as a basis for excluding articles that did not substantially contribute to our understanding of any of the key search terms: "fraction*", "decimal*", or "rational number*" (FDR). The fourth author reviewed the abstracts and main findings of these 40 articles to determine whether they contributed to our understanding of teaching and learning fractions, decimals, and/or rational numbers. Based on the initial reading of these articles, we determined that a threshold of nine occurrences of the presence of either of FDR terms could serve as a reasonable cut-off to include the articles. In other words, if an article used the term fraction, decimal, or rational number nine or more times, it was considered to address pertinent issues related to the teaching and learning of FDR topics. To gain more confidence in this threshold, we selected another random sample of 200 articles to check whether out determination of a threshold of nine or more occurrences of either of the terms, fractions, decimals, or rational numbers, constituted a reasonable criterion that the article was focused on issues related to our study. The second and fourth author read the abstract and findings section of 200 selected articles and concluded that at least nine occurrences of either of the FDR terms in the article were sufficient to ensure that it was sufficiently focused on fractions, decimals, and/or rational numbers. Using this threshold of 9 occurrences led to the deletion of 1364 articles, leaving 802 articles in the main dataset for the final analysis.

Manual Coding Article Characteristics to Capture Research Trends

We coded these 802 articles using MAXQDA to determine the research trends in the target period of 1977 to 2018, with specific attention to research focus, publication year, and type of participants (K-12 students, pre-service teachers, or in-service teachers). Among these articles, we identified 25 that were either book reviews, editorials, or annual research reports (commonly published in JRME). After removing these, we were left with 777 articles for further analysis.

Auto Coding to Capture Discourse Communities

Following Nelson’s (2020) methodology outlined in Pattern Detection Using Human-centered Computational Exploratory Analysis (p. 11), we utilized computer-assisted text analysis techniques to support our research objectives. Leveraging MAXQDA, we auto-coded 777 articles based on our predefined list of 98 terms, established during Stage I. Auto-coding operates through an input-output process, where specific terms serve as inputs, and the system outputs instances of their use within an article. MAXQDA facilitates automatic code assignment to text units, enabling the inclusion of sophisticated wildcards, Boolean operators, and syntactic variations of terms within a single code. We aimed to capture various forms that terms might take; for example, in the case of "Piaget," we accounted for the appearance of "Piagetian" as well.

Previous researchers have raised concerns about the reliability of the lexical search feature due to its reliance on the mechanization of software, which may not always be dependable (Roberts & Wilson, 2002). However, drawing from insights gained in our pilot study, we took steps to mitigate the risk of erroneous coding through mechanized processes. For example, terms such as "conception" and "misconception," or "correct" and "incorrect," were treated as complete search strings to ensure accuracy. To accurately tally instances of the term "conceptions" mechanically, we needed to exclude occurrences of the term "misconceptions." This was achieved by formulating the command as (OR ("conception", "conceptions") NOT (“misconception”, "misconceptions")).

Checking the Auto-Coded Instances

Although the computer generated a list of terms identified as present, this output required human interpretation. Our exploratory work revealed that auto-coding might assign codes not only to the input term but also to irrelevant terms. For example, for the term "scheme," auto-coding also captured "booklet scheme," "coding scheme," and "scheme theory." A similar pattern was observed for the term "intervening," which coded "intervening week" and "intervening factors." Likewise, for terms like "rule," "error," "stage," "procedure," "operation," "reflection," and "association," auto-coding produced false results by capturing related phrases or contexts. To address this issue, we conducted another round of lexical search in MAXQDA, during which we manually removed these falsely coded results.

Recognizing the potential for similar patterns in other coded segments, we chose to explore the diverse meanings of the input terms by analyzing coded excerpts. To achieve this, we compiled a glossary containing definitions provided by the research team alongside illustrative examples sourced from the articles (refer to Table 3). To assess the accuracy of the automated coding process, we randomly sampled 30 excerpts from the coded instances of each term. We specifically examined terms capturing dual or multiple meanings to ensure their interpretation aligned with our intended purpose. For example, the term "operation" was utilized in both cognitive theories and in reference to arithmetical operations. Upon examination, we found that in a sample of 30 randomly selected coded segments, 20 segments interpreted "operation" as referring to arithmetical operations, evident in phrases such as "...students observe that the operation here can be successfully inverted with a multiplication by 9" (Brousseau et al., 2004, p. 16). Conversely, others used "operation" in a constructivist sense, as demonstrated by phrases such as "a splitting operation for composite units" (Olive & Steffe, 2002, p. 424) or "...the use of their iterating operation" (Hackenberg, 2013, p. 555).

Table 3. Examples of Code Description and Code Examples derived from the Literature

Code Description	Example
Conception: Referring to students' strategies or ways of thinking with a constructivist view.	"In this case, Jerry's conception of one-half was one of two equal parts." (Biddlecomb, 2002, p. 184).
Auto-code Input: (OR ("conception", "conceptions") NOT ("misconception", "misconceptions")	"… when she discussed dividing zero by a number brings into question her conception of division" (Levenson, 2013, p. 190).
Misconception: Defining the students' erroneous conceptions leading to inappropriate meanings, prototypical thinking, and over-generalizations.	"…an unawareness that the order of appearance of the numbers was significant. In all, some five specific misconceptions were … " (Bell et al., 1981, p. 405).
Auto-code Input: (OR ("misconception", "misconceptions") NOT ("conception", "conceptions")).	"…we must not only focus on producing fractions, but also on grounded refutations of such misconceptions…" (Steffe, 2001, p. 283).

Similarly, for the term level of units, the search string was used as level of unit and level unit. The MAXQDA captured *one level of unit, two levels of unit, three levels of unit, bottom level unit, higher level units*, and *additional levels of unit.* A further investigation of 30 randomly selected coded segments indicated that 29 segments employed the theoretical meaning of *level of unit*, 20 of which were coded in contexts of one-, two-, or three-levels of units, whereas the rest discussed higher-, mid-, or additional level units. To elaborate, the term *level of units* was commonly used by constructivists (e.g., Steffe, Hackenberg) with a strong theoretical indication to describe students' three-levels of unit coordination. However, other researchers, like Izsák et al. (2012), used terms like *top-level unit, the mid-level unit, the bottom level unit* to describe students' three-level structures as they stated "… shows a second three-level structure in which the unit interval is the top-level unit, fourths are the mid-level unit, and twelfths are the bottom level unit" (p. 404).

For the term *collaboration,* the intended meaning was to capture the social interaction aspect in educational settings. The results showed that the majority (20 out of 30 segments) talked about classroom collaboration, including peer collaboration (Irwin, 2001); classroom collaborative dialogue (Prediger & Wessel, 2013); collaborative interaction with the teacher (Sáenz-Ludlow, 1995); collaboration with more capable peers (Nabors, 2003); collaborative group discussions (Kaminski, 2002). Meanwhile, 10 segments disclosed professional collaborations of conducting the studies. For instance, collaborative research study (Hodgen et al., 2010); educator collaborative groups (Lewis & Perry, 2017); productive professional collaboration (Kofi-Davis, 2017); collaboration in preparation for task implementation (Thanheiser et al., 2016); and future collaboration on teaching

(Stacey et al., 2001).

In the same analysis, the term labeled as *shift* was deleted due to the multiple ways it was used by various researchers in the coded segments (28/30). Most of the occurrences of the term shift indicated a general change in one's way/style of thinking or work, rather than any specific theoretical construct. On similar grounds, other terms were interchanged with each other (e.g., procedural fluency with procedural knowledge, reflected abstraction with reflective abstraction, interiorize with interiorization). By the end of this filtering process, we were left with a list of 85 terms.

This step not only deepened our understanding of the mathematics education field but also facilitated interpretation and identification of patterns among the terms. Additionally, it provided insight into distinct discourse communities within the field, which Savich et al. (under-preparation) have referred to as silos. They elaborate that "research produced through one silo may be theoretically incommensurable with the research produced through a different silo [e.g., use of the term level of units as described above], so that even if the results of one study seem to harmonize with the results of another, they do not form a coherent whole because they address different phenomena examined from ontologically and epistemologically incompatible perspectives. Siloing is good for specialization, but it can leave the practitioner community without any sense of "the research" as it stands due to the incommensurable findings and frameworks" (Savich et al., under-preparation, p. n.d.). From a practical standpoint, these checks along the way provided confirmation that the auto-coding process of MAXQDA had captured segments related to our research purposes.

Preparing Data to Examine the Corpus of Literature

In Nelson's (2020) methodology, another crucial step involves confirming whether the identified patterns remain consistent throughout the dataset using computational techniques. In this stage, each individual article serves as the unit of analysis (see Table 4).

Table 4. Journals included in the Literature Review Final Step

Journal name	Selected articles
Journal of Research in Mathematics Education (JRME)	108
Educational Studies in Mathematics (ESM)	157
The Journal of Mathematical Behavior (JMB)	106
ZDM-The International Journal on Mathematics Education (ZDM)	83
Journal of Mathematics Teacher Education (JMTE)	75
Mathematical Thinking and Learning (MTL)	17
Research in Mathematics Education (RME)	15
Mathematics Education Research Journal (MERJ)	61
International Journal of Science and Mathematics Education (IJSME)	47
International Journal of Mathematical Education in Science and Technology (IJMEST)	56

We extracted the frequency of occurrence of each term in an article using MAXQDA and utilized this data to dichotomize the information. To do this, we established a criterion: if a term appeared more frequently in an article than the median value of the non-zero frequency of that specific term across all articles, we assigned the value of "1" to that code in that article. For example, consider the term "discourse," which appeared between 0 and 172 times across the articles. We computed the median of all non-zero values, resulting in a value of 2. Consequently, we re-coded all numerical count values that were less than 2 as 0, indicating absence or infrequent use, and assigned a value of 1 to those equal to or greater than 2, indicating more frequent use.

After dichotomizing each term, we established a criterion for excluding terms based on their overall occurrence in the articles. If the count for a term fell below 5% of the total articles, we decided to remove that term from consideration. For example, consider the term "aptitude." The count of this term across all articles was 22, which was lower than 5% of the total articles (approximately 39). Consequently, we discarded the term from the analysis. Sparse usage of terms poses a challenge for Exploratory Factor Analysis (EFA) due to its sensitivity to low-frequency and sparse counts. As a result, we deleted 13 terms due to their infrequent occurrence. Additionally, we removed 52 articles from the dataset as they were not coded with any terms. This refinement process resulted in a final dataset consisting of 725 articles and 72 codes. Figure 1 provides a visual graphic to show how articles were screened at every step. Figure 2 explains the inclusion criteria for terms.

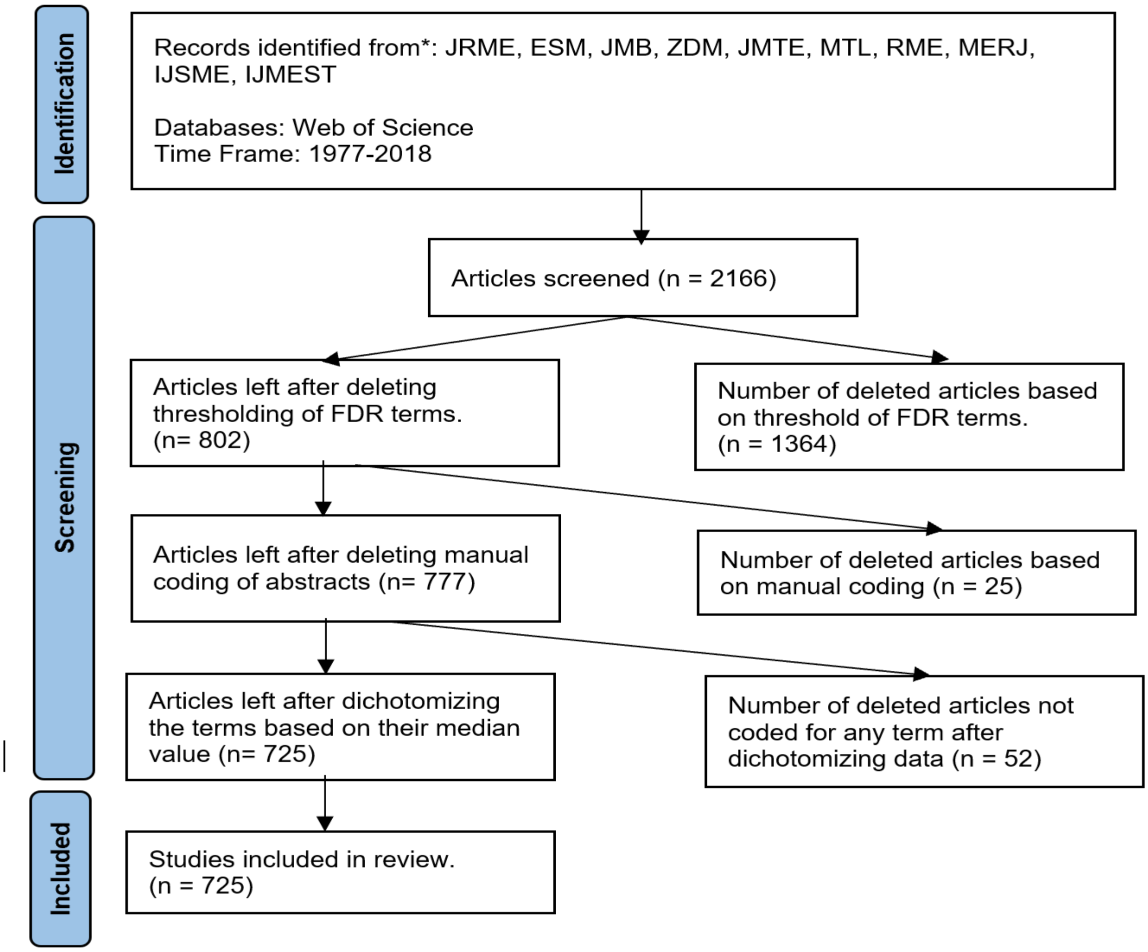

Figure 1. PRISMA Flow Diagram to Show How Articles were Selected in Stage 2

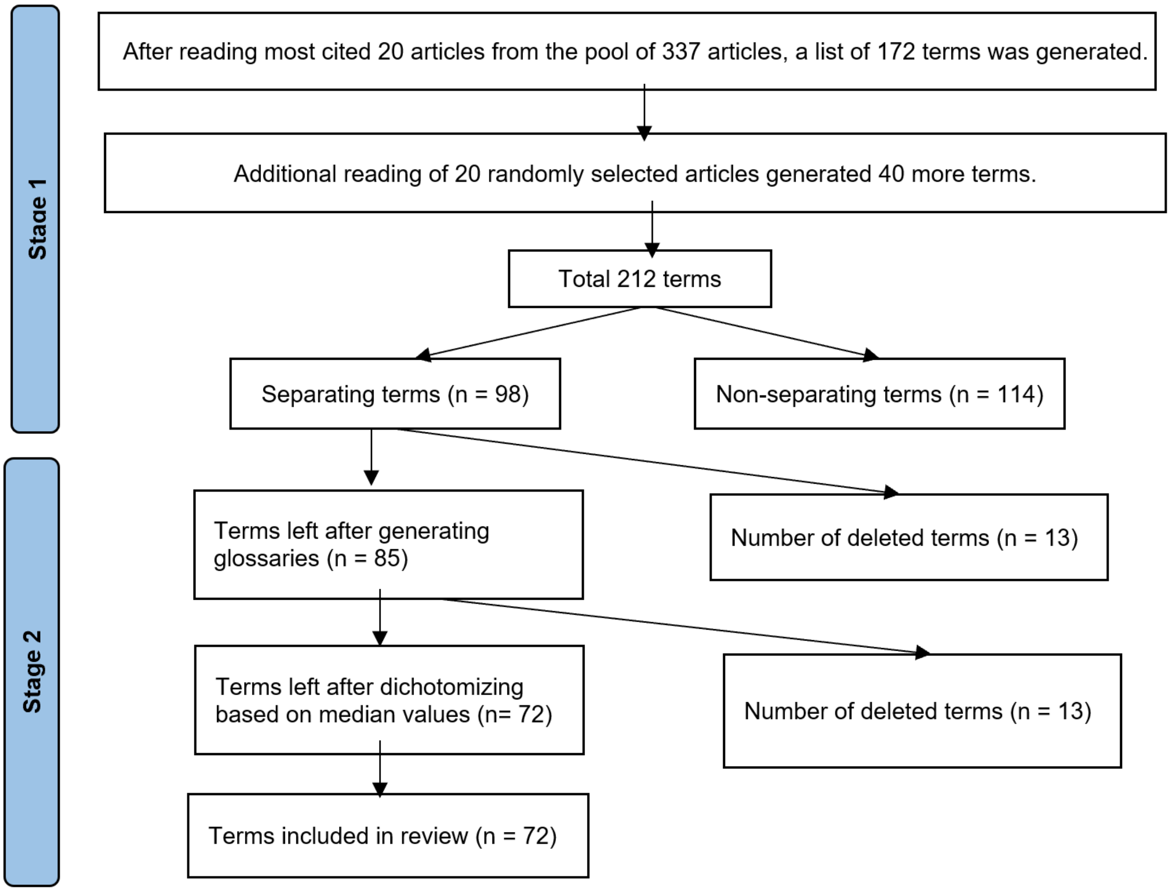

Figure 2. Inclusion Criteria for Terms

Quick Glimpse of Publication Trends

To gain an initial understanding of the trends in published literature in mathematics education spanning four decades, we delved into the number of articles published between 1977 and 2018 (Figure 3), as well as the distribution across each journal (Figure 4). Figure 3 illustrates a steady increase overtime, with the greatest number of articles, 436, published in the last half-decade of our study, marking a significant 96% increase compared to the preceding years (2007-2012). This figure tells a compelling story of how the research landscape can evolve, stabilizing at times and then experiencing periods of rapid growth, reflecting changes in the field's maturity, popularity, or overall interest.

Figure 4 provides a visual representation of how each of the top-tier journals has contributed to the development of ideas related to fractions, decimals, and rational numbers (FDR). There is a general trend of increasing publication frequency across all journals over the years. Particularly, the period from 2007-2012 and beyond shows a significant jump in the number of articles published. This indicates growing research activity or greater acceptance of articles across these journals in the most recent periods. Initially, ESM published most articles on these topics, but over time, ZDM and JMB also made substantial contributions with equal rigor.

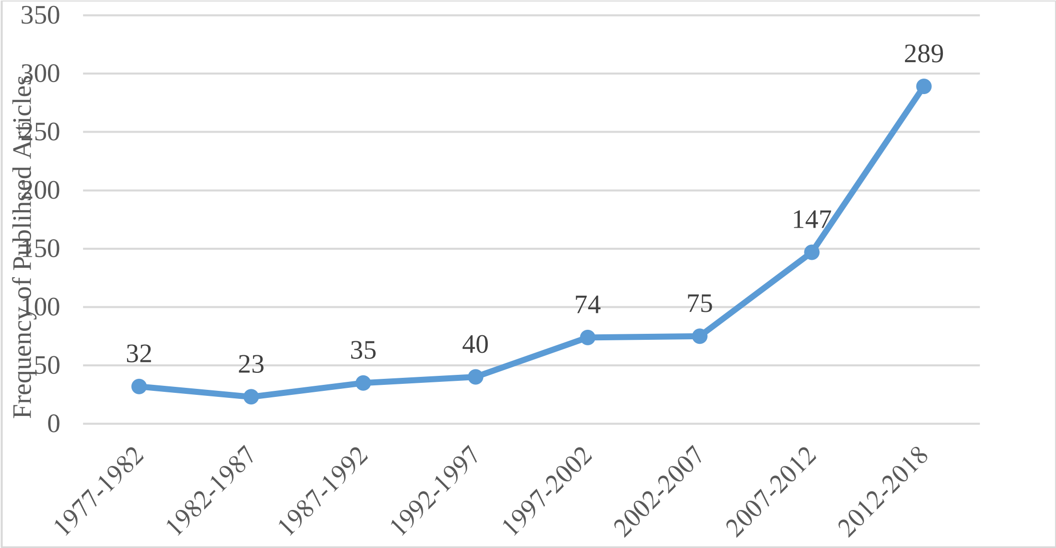

Figure 3. Frequency of FDR Articles Published in Top Ten Journals during the Last Four Decades

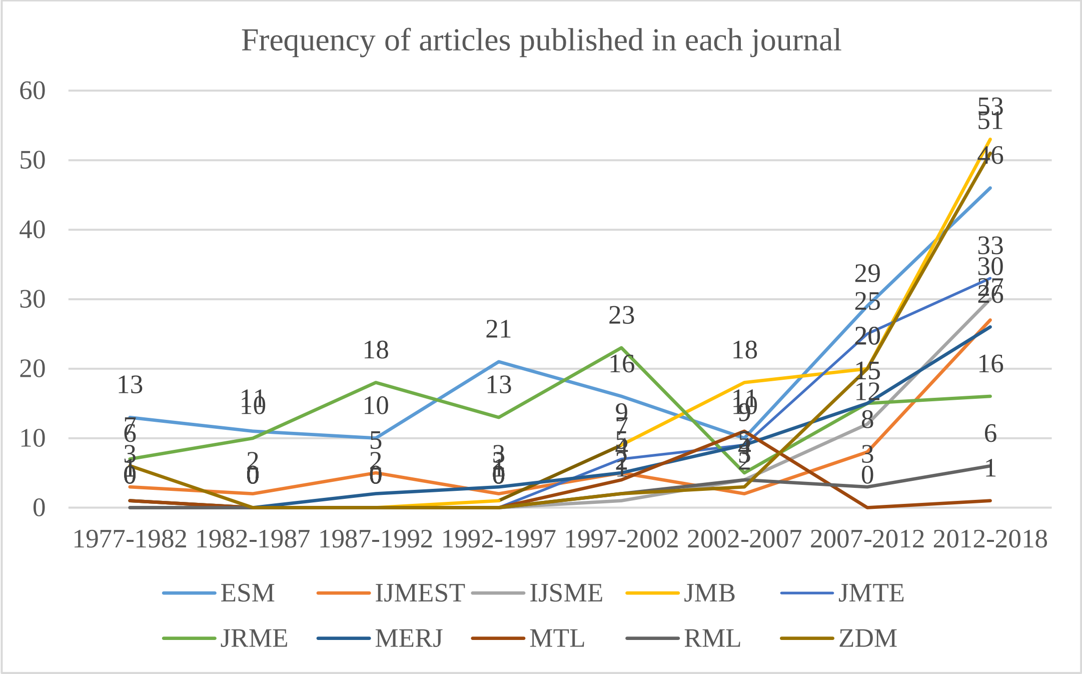

Figure 4. Frequency of FDR Articles Published in Each Journal

Analysis

As we delve into our analysis, we want to recall our primary research inquiry: *How have discourse communities within mathematics education explored and shaped the understanding of fractions, decimals, or rational numbers over the past four decades?* We employed Exploratory Factor Analysis (EFA) to address our main research inquiry. EFA is employed to identify latent variables, or underlying factors, that represent discourse communities within the field of mathematics education. These latent variables are not directly observable but are inferred from patterns of correlations among observed variables, in this case, the frequency and co-occurrence of specific terminologies within our corpus of literature. Each factor derived from the EFA represents a distinct discourse community, characterized by a unique cluster of interrelated terms that signify shared concepts, methodologies, or thematic focuses. This analytical approach helps us to systematically delineate and describe the implicit, often complex, structures within the academic dialogues surrounding fractions, decimals, or rational numbers, thus, providing a nuanced understanding of the landscape of research in mathematics education.

Prior to conducting the EFA analysis, we assessed the correlations among terms to gauge their interrelatedness. EFA enabled us to discern patterns across terms by clustering them statistically, thereby unveiling overarching themes within the dataset. Analogous to the application of EFA in assessment development, we regarded each article as an examinee and each separating term as an assessment item. Through EFA, we refined the concept of terms in our analysis—these terms coalesce into identifiable factors in the EFA, which we delineate in the findings section as distinctive components of the vocabulary of a specific discourse community. We utilized the diagonally weighted least squares with mean and variance adjusted (WLSMV) extraction method via Mplus (Muthén & Muthén, 2012; version 7.11 for Mac) to conduct the EFA with three and six factors. Given the expected correlation among terms, we employed an oblimin rotation (Tabachnick & Fidell, 2013). We evaluated the structures of all 72 codes to ensure that loadings exceeded .300, meeting our criterion for inclusion.

Results from EFA Analysis

Prior to conducting the exploratory factor analysis (EFA), we assessed the correlations between dichotomized terms to gauge their interrelationships. Tetrachoric correlation, a method suitable for binary data, assumes the presence of an underlying continuous scale and normal distribution of underlying variables (Juras & Pasarić, 2006; Roscino & Pollice, 2006). While meeting the normality assumption was challenging due to the skewed nature of the data, we calculated tetrachoric correlations between the variables based on the unimodal skewed data (Uebersax, 2006). All the terms seemed to have varied degrees of correlation. For instance, some coded pairs showed no correlation (value of 0), e.g., *classroom discussion and aptitude* or *assimilation and dialog,* whereas others had positive correlations, e.g., *Piaget and reflected abstraction* (value of 0.66) or negative correlations, e.g., *correct and activity theory* (value of -0.43).

We employed the maximum likelihood extraction method to determine the number of factors from the 72 terms. Several approaches have been used to determine the optimal number of factors, including eigenvalues greater than 1 (Guttman, 1954), parallel analysis (Horn, 1965), and identifying the major elbow on the scree plot (Cattell, 1966). Based on theoretical considerations, we anticipated identifying 3-, 4-, 5-, and 6-factor models from the EFA analyses as there are distinct schools of thought within mathematics education. Historically, these schools of thought have revolved around key pedagogical and cognitive frameworks, such as constructivism, social constructivism, realism, and behaviorism. Each of these frameworks offers a different perspective on how mathematical concepts should be taught and understood. Our EFA aimed to capture these underlying theoretical distinctions by identifying latent variables that correspond to these diverse educational philosophies. We hypothesized that each factor would represent a cluster of research and discussion closely aligned with one of these foundational perspectives, thereby revealing the complex interplay of ideas that shapes research in mathematics education. Our preliminary examination suggested the likelihood of identifying at least three discourse communities. Consequently, we conducted EFA models for all four scenarios and assessed their fit indices, as presented in Table 5. Below, we present the 3-, 4-, 5-, and 6-factor models, with each factor representing a distinct discourse community. In the results section, we focus on the findings from the 3- and 6-factor models, as they offer compelling insights into the discourse communities involved in rational number research within mathematics education.

Table 5. Fit Statistics for the EFA Models

Model	df	χ^2	RMSEA	90% C.I.	CFI	TLI	SRMR
3-factor	1272	1808.61	.025	[.022, .027]	.820	.797	.096
4-factor	1221	1578.04	.021	[.018, .024]	.880	.859	.087
5-factor	1171	1436.45	.018	[.015, .021]	.911	.891	.079
6-factor	1122	1320.06	.016	[.012, .020]	.933	.915	.073

Discussion

Following Nelson's (2020) methodology, we sought to discern patterns among the terms to interpret the findings of the exploratory factor analysis (EFA) within the context of mathematics education. The results are presented as groupings of terms, akin to a taxonomy. Table 1 and 2 (Appendix) displays the 3-factor, 4-factor, 5-factor, and 6-factor models, reporting only factor loadings equal to or greater than 0.30, as smaller loadings are typically deemed insignificant (Kline, 2002; Tabachnick & Fidell, 2001). In Figure 5, we provide a simplified diagram illustrating how the groupings of terms evolve from the 3-factor model to the 6-factor model, resembling the process of mitosis. We chose to depict only those terms that were loaded in each of the four models, enabling readers to discern the shifts in term groupings as the dimensionality (number of discourse communities) accommodated by the EFA analysis increased.

In Figure 5, we employ the metaphor of mitosis to depict the results of the exploratory factor analysis (EFA). The metaphorical narrative unfolds as the EFA process progresses, allowing for the separation of initial factors into more distinct entities. We envision that with the inclusion of additional terms, improved algorithms, and a larger dataset, more subtle distinctions could be discerned. This projection underscores how our methods and interpretation serve as a form of systematic and accountable theoretical bricolage on a broader scale, offering a means of reading-without-reading.

We noticed that the codes reveal an interesting pattern based on the number of factors extracted at each iteration of the EFA analysis (F_m^n means the n^th^ factor in the m-factor model). The three-factor model showed that each factor (F_3^1, F_3^2, and F_3^3) had 20, 15, and 6 terms, respectively. For the four-factor model, the three factors (F_4^1, $F_4^2$and F_4^4) maintained approximately the same structure (F_3^1, F_3^2, and F_3^3); however, the structure of the fourth factor was based on the terms from some of the existing factors in the 3-factor model or the un-used terms (i.e., for which factor loadings were smaller than .30 in magnitude for a three-factor model). What intrigued us as researchers is that the codes conveying similar meaning remained within their own factors (e.g., F_3^1, F_4^1, F_5^1, and F_6^1 captured a similar set of words, *anticipation, assimilation, conception, constructivism, in-activity, informal* *knowledge, interiorization, operator, perturbation, Piagetian, radical constructivism, reorganization hypothesis, scheme, scheme theory*). For the F_3^2, the codes capture the related to Vygotskian theory, e.g., what should be done within a classroom? (e.g., argumentation, collaboration, classroom discussion, dialog, conversation, discourse). All these codes captured ideas related to classroom communication or classroom norms. This pattern of the codes remained similar until F_6^3.

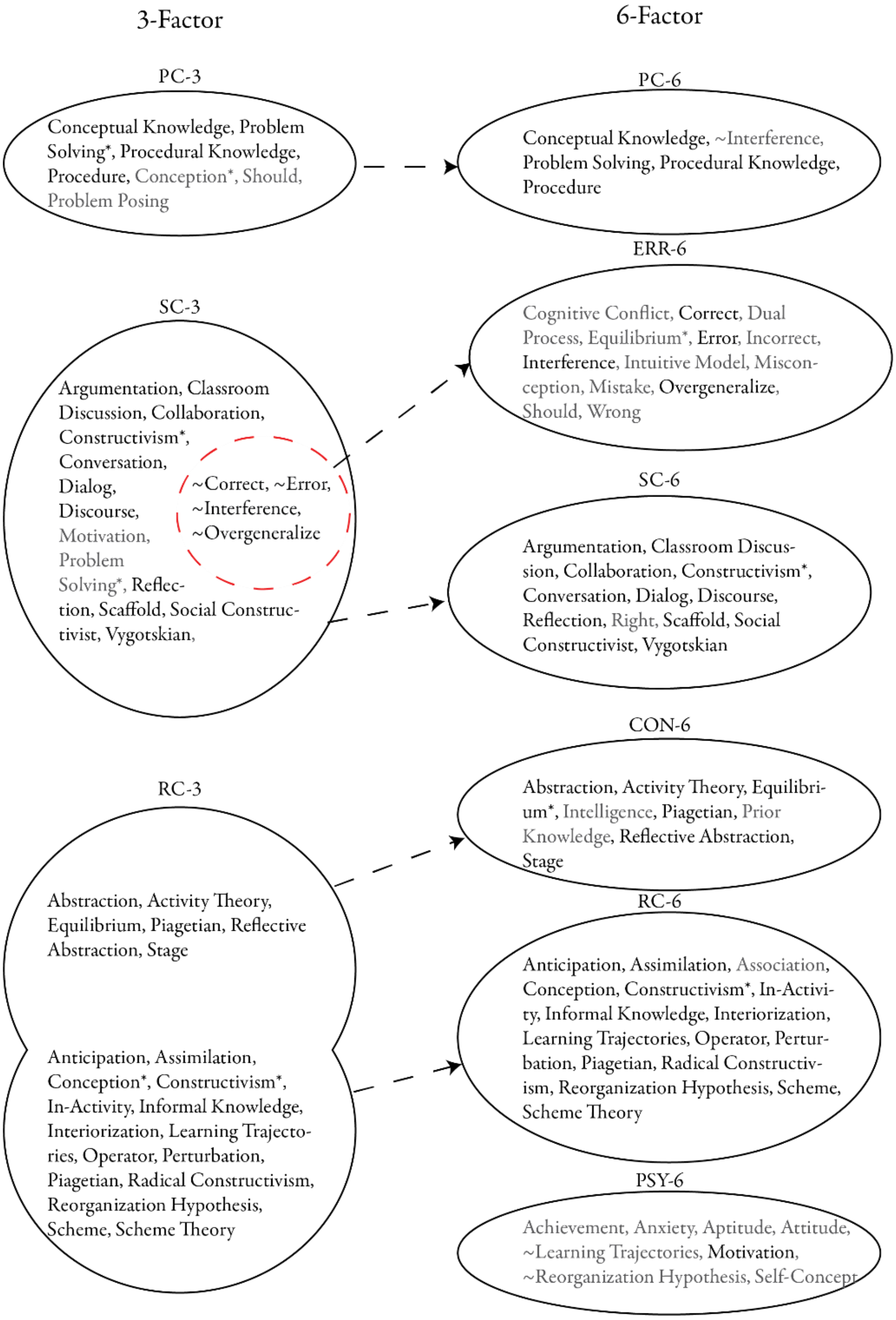

Figure 5. Story Setting from EFA Analysis

Note. This figure introduces a temporal metaphor of mitosis to a simplified collection of terms in the 3-factor and 6-factor models. Negative loadings are preceded with “~,” dual loadings are non-separating and are indicated with “*,” and greyed out terms do not follow the pattern of the other terms in their groups.

A similar trend was observed for some other factors (e.g., F_3^3, F_4^4, F_5^5, and F_6^5). However, with additional factors, these words could re-group to convey a more nuanced understanding of the topic. For instance, F_4^3captured the words related to ideas of students' erroneous patterns, *error, incorrect, misconception, mistake, overgeneralize, wrong*, indicating what teachers need to be cautious of while working with students. And this factor split into two factors in a five-factor model, in which F_5^3 captured terms like *aptitude, equilibrium, intelligence*, or *stage*, whereas F_5^4 captured other terms. We tell the story in these results simply by naming the factors in the 3-factor and 6-factor models. The articles are grouped together into factors based on a shared lexicon. We do not claim that these factors constitute discourse communities in mathematics education; instead, our claim is much more modest. Instead, we claim that the factors (based on lexical clusters co-occurring in mathematics education articles) tell us something important about the role of language in discourse communities that exist in mathematics education. For the procedural/conceptual debate concerning the 3-factor model (PC-3), we name the discourse communities "social constructivism" (SC-3) and "radical constructivism" (RC-3). The procedural/conceptual factor's designation refers to the long-standing distinction between those knowledge types (i.e., Crooks & Alibali, 2014; Hiebert, 1988). PC-3 is also remarkably stable, so we name it its corollary PC-6.

The factor SC-3 is so named because its key terms, such as *dialog* and *discourse,* signify social reasoning and productive interaction, and it corresponds to an actual discourse community, "social constructivism." Most of these SC-3 terms also carry over to the 6-factor model. The negative loadings in SC-3 are on terms that suggest codifying certain utterances as correct or incorrect, indicating that researchers in SC-3 community have less interest in such definitive judgments than those in other discourse communities. The terms which are negatively loading in SC-3 cluster together in what we call ERR-6, referring to the core terms *error, wrong, mistake, misconception, incorrect*. However, we note that this factor might be named in association with the discourse community "cognitive psychology" due to the presence of the terms *cognitive conflict, dual process*.

RC-3 includes the term with which this discourse community, "radical constructivism," is named. RC-3 also contains several related terms such as *scheme, perturbation, Piaget*. The bulk of RC-3 maps to RC-6, suggesting a stable core of terms used by the "radical constructivist" discourse community. However, the 6-factor model seems to split into two distinct factors, one (CON-6) that is arguably more associated with generic constructivism than with radical constructivism" (RC-6). CON-6 includes terms like *activity theory*, which is a Vygotskian idea (i.e., Roth, 2014), but it also includes *Piagetian* and *reflective abstraction*, which suggests a more general constructivist discourse community.

With respect to PSY-6, which is named in association with the discourse community "cognitive psychology,". In this factor we have the terms {Achievement, Anxiety, Aptitude, Attitude, ~Learning Trajectories, Motivation, ~Reorganization Hypothesis, Self-Concept}. Researchers in this vein discuss such topics as the relationship between students' self-concepts and their ability to achieve mathematics correctly (e.g., Pietsch et al., 2003). There is also a fair amount of literature discussing the negative effects of mathematics anxiety on performance indicators. The terms *motivation* and *attitude* also suggest the view that psychological states partially explain differences in student performance.

These names tell the story that mathematics education research on teaching rational numbers comprises distinct discourses about procedural versus conceptual knowledge, social constructivism, and radical constructivism. One noteworthy feature of the diagram is that what coheres for SC-3 and PC-3 in the 3-factor model also coheres in the 6-factor model. RC-3, on the other hand, deviates in that terms associated with the discourse community "radical constructivism" appear in both RC-6 and CON-6. This phenomenon suggests possible future avenues of exploration making use of more sophisticated concepts from graph theory, such as clustering coefficients, which measure how strongly nodes (terms, in this context) are connected to one another (i.e., Soffer& Vazquez, 2005). With the data in Table 1 and Table 2 (Appendix), we invite readers to challenge these names and pursue other possible story lines.

Our methodology was specifically designed to meet the unique demands of this study, focusing on the nuanced interpretation of language within mathematics education literature. While this approach was customized for our research objectives, it is based on principles that could be beneficial for others exploring similar textual analyses. Following Nelson's (2020) guidance, we believe that our methodological framework can provide a robust foundation for researchers aiming to derive meaningful insights from complex academic texts. Our exploratory way to uncover novel, intriguing, and practical patterns within the datasets. The summarized steps of knowledge discovery we took are – Stage 1 - i. finalizing the data domain: our project specifically focused on the discourse surrounding FDR within mathematics education, hence, we narrowed down the selection to specific journals; ii. processing the data: we extracted metadata such as journal name, year, volume, title, author, abstract, keywords, and more from each article, which was meticulously recorded and organized in a Microsoft Excel spreadsheet; iii. determining the parameters for data mining: we employed computational algorithms to identify patterns and clusters of terms, which served as markers to pinpoint specific discourse communities prevalent within the field; iv. evaluating and deploying patterns: once the patterns were identified, evaluation involved interpreting these patterns to understand their implications in the broader context of mathematics education research. Stage 2 - we discovered clustered terms around certain themes which helped us to delineate distinct discourse communities. These findings were then deployed to inform and enhance our understanding of the research landscape using technological advancements, contributing insights into how various concepts and methodologies have evolved over time within the field.

Limitations and Future Work

While this research was exploratory in nature, designed to navigate the extensive volume of literature and integrate technological tools effectively, we believe it has successfully uncovered diverse theoretical perspectives within the field of mathematics education. Initiated in 2018, our methodology was crafted to reflect the best practices and technologies available at the time. As advancements like OpenAI become more prevalent in academic settings, they open new possibilities for methodological evolution. However, our approach, combining both quantitative and qualitative methodologies, was meticulously chosen and executed by our team of mathematics educators, each bringing a rich diversity of expertise to the project. We appreciate the positive recognition of our efforts to delineate the various schools of thought that inform mathematics education research.

In terms of limitations, during the EFA, we did not utilize the entire lexicon of each article because it was necessary to remove non-separating terms from the lexical analysis to ensure convergence of the statistical models. Therefore, claims of sufficiency cannot be upheld in practical terms due to the exclusion of certain words. Furthermore, our sampling of articles to form our initial sets of terms was not exhaustive (see Methods section), and alternative choices of terms might have yielded different results. To maintain theoretical coherence, it's necessary to posit a class of non-separating and implicit terms to address the gaps left by the empirical methods we employed. Another potential limitation is that our examination focused solely on trends within the top ten journals in mathematics education, excluding information from published conference proceedings, dissertations, or practitioner journals. It's conceivable that these clusters might exhibit different trends had we included information from all published resources in mathematics education over the four decades of the study period. Additionally, regarding the keywords used in the EFA analysis, we acknowledge that the list is not exhaustive. However, we believe it serves as a productive starting point for exploring trends in one of the most researched topics in mathematics education literature: rational numbers. While the final stage of Nelson’s (2020) approach involves confirming identified patterns rather than establishing definitive or causal relationships in the text, we have not yet completed this final stage in our analysis. This step could potentially be undertaken in the future by including information from other published literature in mathematics education and replicating a similar analytic process.

Conclusion

Through an examination of published work in mathematics education journals spanning four decades, we employed statistical methods and automated lexical analysis to gain insights into prevailing beliefs and values within various research circles in math education. Our approach, a blend of manual coding and computational analyses, represents a novel form of reading-without-reading, extending quantitative methods from the humanities into the realm of mathematics education. While this method may invite criticism for its reliance on terms divorced from their original contexts, our process involves manual checks of a subset of texts. This enabled us to glean meaning from literature and contextualize our findings within the broader discourse, mitigating some concerns associated with automated processes alone.

As an example of leveraging the affordances of existing technologies, this study might offer new insights into assessing the state of any field, e.g., by examining the shifting landscapes within the published literature. The computational capability we employed allowed us to identify the presence of terms in the published literature, and the EFA revealed patterns in ways which are not readily discernable for human readers. This approach thus facilitates finding novel ways to perceive and analyze data that may reveal interesting patterns and support inferences and interpretations that enhance understanding of the ongoing development of a field. Furthermore, using sophisticated software enhanced the reproducibility of the findings as well as helping us to re-engage with the data in meaningful ways. Some of our design decisions entailed the involvement of human judgment in the analysis, e.g., selecting or removing terms and interpreting the meaning of each cluster as given by the EFA analysis, which necessitated examining excerpts in the manuscripts in which the terms to generate the glossary were used. The component of human judgment in this methodology might be perceived as a limitation by some, but we also believe that relying solely on computational results might not be sufficient. The presence of the researcher's expertise allowed us to double-check the findings and make informed inferences. We believe that this hybrid approach can be refined in the future to attain a holistic sense of the range of meanings of terms in large datasets.

It is also important to clarify how our methods—and in particular the way in which we generated codes and analyzed them—limit how the results can be interpreted. We are not implying, for example, that only radical constructivists (RC-3) use the term *stages*. The pragmatic contexts in which we deployed terms such as *stages* are articulated in our methods section, and terms in our glossary are divorced from their original contexts. Any papers that include the term *stages* would contribute to the "radical constructivist" factor even though the term is not exclusive to that community. The results from the EFA analysis seem sensible and give us confidence about the methodology we used. In the future, it would be interesting to examine how the use of the terms shifted over time to show the evolving nature of the mathematics education field. We also believe that similar techniques could be used to explore the patterns of other contexts within mathematics education.

In the discussion section, we shared one way of identifying discourse communities in mathematics education, but the variety of patterns that might be discerned in the given EFA results suggest potential to develop other complementary narratives. Instead of claiming this is the sole interpretation, we offer this possibility in service of a larger goal. By reporting our work on reading-without-reading, we believe that we have shed light on an approach with potential for advancing ways of conducting literature reviews. This approach has promised both the research community and other stakeholders who might otherwise get lost in the breadth and tangle of incommensurable frameworks and findings in mathematics education research by leveraging computational, quantitative techniques that are validated with selective textual analysis.

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References

Biddlecomb, B. D. (2002). Numerical knowledge as enabling and constraining fraction knowledge: An example of the reorganization hypothesis. *The Journal of Mathematical Behavior, 21*(2), 167-190.

Bell, A., Swan, M., & Taylor, G. (1981). Choice of operation in verbal problems with decimal numbers. *Educational studies in Mathematics, 12*(4), 399-420.

Brousseau, G., Brousseau, N., & Warfield, V. (2004). Rationals and decimals as required in the school curriculum: Part 1: Rationals as measurement. *The Journal of Mathematical Behavior, 23*(1), 1-20.

Cattell, R. B. (1966). The scree test for the number of factors. *Multivariate Behavioral Research, 1*, 245-276.

Crooks, N. M., & Alibali, M. W. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental review*, *34*(4), 344-377.

Guttman, L. (1954). Some necessary conditions for common-factor analysis. Psychometrika, 19(2), 149-161.

Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior*, *32*(3), 538-563.

Hiebert, J. (1988). A theory of developing competence with written mathematical symbols. *Educational studies in mathematics*, *19*(3), 333-355.

Horn, J. L. (1965). A rationale and test for the number of factors in factor analysis. *Psychometrika, 30*, 179-185.

Izsák, A., Jacobson, E., de Araujo, Z., & Orrill, C. H. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education*, *43*(4), 391-427.

Juras, J., & Pasarić, Z. (2006). Application of tetrachoric and polychoric correlation coefficients to forecast verification. *Geofizika*, *23*(1), 59-82.

Kaminski, E. (2002). Promoting mathematical understanding: Number sense in action. *Mathematics Education Research Journal*, *14*(2), 133-149.

Kline, R. B. (2002). *Principles and practice of structural equation modeling*. Guilford publications.

Kofi-Davis, E. (2017). A Three-Tier Teaching Model for Teaching Mathematics in Context. *For the Learning of Mathematics*, *37*(2), 14-15.

Levenson, E. (2013). Exploring one student's explanations at different ages: the case of Sharon. *Educational Studies in Mathematics*, *83*(2), 181-203.

Lewis, C., & Perry, R. (2017). Lesson study to scale up research-based knowledge: A randomized, controlled trial of fractions learning. *Journal for research in mathematics education*, *48*(3), 261-299.

Muthén, L. K., & Muthén, B. O. (2012). *Mplus user's guide* (7th ed.). Los Angeles, CA: Muthen & Muthen.

Nabors, W. K. (2003). From fractions to proportional reasoning: a cognitive schemes of operation approach. *The Journal of Mathematical Behavior*, *22*(2), 133-179.

Nelson, L. K. (2020). Computational grounded theory: A methodological framework. *Sociological Methods & Research*, *49*(1), 3-42.

Nivens, R. A., & Otten, S. (2017). Assessing Journal Quality in Mathematics Education. *Journal for Research in Mathematics Education, 48*(4), 328–368.

Olive, J., & Steffe, L. P. (2002). The construction of an iterative fractional scheme: The case of Joe. *The Journal of Mathematical Behavior*, *20*(4), 413-437.

Pietsch, J., Walker, R., & Chapman, E. (2003). The relationship among self-concept, self-efficacy, and performance in mathematics during secondary school. *Journal of educational psychology*, *95*(3), 589.

Prediger, S., & Wessel, L. (2013). Fostering German-language learners' constructions of meanings for fractions—design and effects of a language-and mathematics-integrated intervention. *Mathematics Education Research Journal*, *25*(3), 435-456.

Roberts, K. A., & Wilson, R. W. (2002). ICT and the research process: Issues around the compatibility of technology with qualitative data analysis. *Qualitative Social Research, 3*(2), 83-94.

Roth, W. M. (2014). Science language Wanted Alive: Through the dialectical/dialogical lens of Vygotsky and the Bakhtin circle. *Journal of Research in Science Teaching, 51*(8), 1049-1083.

Roscino, A., & Pollice, A. (2006). A generalization of the polychoric correlation coefficient. In *Data analysis, classification and the forward search* (pp. 135-142). Springer, Berlin, Heidelberg.

Saenz-Ludlow, A. (1995). Ann's fraction schemes. *Educational Studies in Mathematics*, *28*(2), 101-132.

Savich, T. M., Jacobson, E., Kaur Bharaj, P., Liu, J., & Eker, A. (under-preparation). Vocabularies of Mathematics Education.

Soffer, S. N., & Vazquez, A. (2005). Network Clustering Coefficient Without Degree Correlation Biases. *Physics Review E* , *71*, 1-4.

Stacey, K., Helme, S., Steinle, V., Baturo, A., Irwin, K., & Bana, J. (2001). Preservice teachers' knowledge of difficulties in decimal numeration. *Journal of mathematics teacher education*, *4*(3), 205-225.

Steffe, L. P. (2001). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behavior*, *20*(3), 267-307.

Tabachnick, B. G. & Fidell, L. S. (2013). *Using Multivariate Statistics*, Sixth Edition. Upper Saddle River, NJ: Pearson Education, Inc.

Thanheiser, E., Olanoff, D., Hillen, A., Feldman, Z., Tobias, J. M., & Welder, R. M. (2016). Reflective analysis as a tool for task redesign: The case of prospective elementary teachers solving and posing fraction comparison problems. *Journal of Mathematics Teacher Education*, *19*(2-3), 123-148.

Uebersax, J. S. (2006). Introduction to the tetrachoric and polychoric correlation coefficients. *Obtenido de http://www. john-uebersax. com/stat/tetra. htm*.

VERBI Software. (2016). MAXQDA Analytics Pro [Computer Programme]. Berlin: Germany: VERBI.

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Appendix.

Table 1. Factor Analysis for 3-factor and 4-factor Model

	3-Factor			4-Factor			
Terms	F_3^1	F_3^2	F_3^3	F_4^1	F_4^2	F_4^3	F_4^4
Abstraction	0.64			0.62			
Achievement							
Activity Theory	0.31					-0.38	
Anticipation	0.49			0.48			
Anxiety							
Aptitude							
Argumentation		0.37			0.42		
Assimilation	0.76			0.76			
Association							
Attitude							
Classroom Discussion		0.44			0.4		
Cognitive Conflict						0.46	
Collaboration		0.33			0.3		
Conception	0.49		0.3	0.51			
Conceptual Knowledge			0.69				0.75
Constructivism	0.65	0.33		0.65	0.39		
Conversation		0.6			0.61		
Correct		-0.3				0.52	
Dialog		0.51			0.61		
Discourse		0.71			0.63		
Dual Process						0.38	
Equilibrium	0.42			0.45			
Error		-0.33				0.47	
In-Activity	0.38			0.36			
Incorrect						0.54	
Informal Knowledge	0.36			0.35			
Intelligence							
Interference		-0.41				0.37	
Interiorization	0.79			0.78			
Intervening					0.33		
Intuitive Model						0.37	
Learning Trajectories	0.53			0.5		-0.31	
Misconception						0.44	
Mistake						0.46	
Motivation		0.34			0.35		

	3-Factor			4-Factor			
Terms	F_3^1	F_3^2	F_3^3	F_4^1	F_4^2	F_4^3	F_4^4
Operator	0.47			0.5			
Overgeneralize		-0.45				0.44	
Perturbation	0.73			0.75			
Piagetian	0.74			0.73			
Prior Knowledge							
Problem Posing			0.35				0.31
Problem Solving		0.33	0.39		0.34		0.37
Procedural Knowledge			0.84				0.86
Procedure			0.6				0.59
Radical Constructivism	0.79			0.8			
Reflection		0.35			0.36		
Reflective Abstraction	0.74			0.72			
Reorganization	0.81			0.81			
Hypothesis							
Right							
Scaff		0.41			0.32		
Scheme	0.77			0.77			
Scheme Theory	0.79			0.8			
Self-Concept							
Should			0.35			0.31	
Social Constructivism		0.47			0.52		
Stage	0.31			0.3			
Vygotskian		0.43			0.44		
Wrong						0.4	

Table 2. Factor Analysis for 5-factor and 6-factor Model

	5-Factor					6-Factor					
Terms	F_5^1	F_5^2	F_5^3	F_5^4	F_5^5	F_6^1	F_6^2	F_6^3	F_6^4	F_6^5	F_6^6
Abstraction			0.57					0.56			
Achievement	-0.31										0.31
Activity Theory			0.39	-0.31				0.39			
Anticipation	0.32					0.35					
Anxiety											0.46
Aptitude	-0.3		0.37								0.43
Argumentation		0.41					0.45				
Assimilation	0.64					0.66					
Association	0.44					0.45					
Attitude											0.59
Classroom Discussion		0.42					0.52				
Cognitive Conflict				0.52					0.48		
Collaboration		0.31					0.33				
Conception	0.38					0.37					
Conceptual Knowledge					0.73					0.75	
Constructivism	0.55	0.41				0.6	0.35				
Conversation		0.62					0.59				
Correct				0.44					0.53		
Dialog		0.61					0.57				
Discourse		0.69					0.77				
Dual Process				0.44					0.36		
Equilibrium			0.41	0.38				0.42	0.3		
Error				0.43					0.45		
In-Activity	0.41					0.4					
Incorrect			-0.32	0.44					0.52		
Informal Knowledge	0.38					0.35					
Intelligence			0.35					0.35			
Interference		-0.32		0.46	-0.32				0.45	-0.33	
Interiorization	0.81					0.79					
Intervening		0.3		0.3							
Intuitive Model				0.33					0.33		
Learning Trajectories	0.39			-0.32		0.31					-0.37

	5-Factor					6-Factor					
Terms	F_5^1	F_5^2	F_5^3	F_5^4	F_5^5	F_6^1	F_6^2	F_6^3	F_6^4	F_6^5	F_6^6
Misconception				0.42					0.46		
Mistake				0.39					0.46		
Motivation		0.35									0.35
Operator	0.65					0.6					
Overgeneralize				0.38					0.48		
Perturbation	0.66					0.7					
Piagetian	0.50		0.37			0.51		0.36			
Prior Knowledge								0.32			
Problem Posing											
Problem Solving		0.33			0.34					0.32	
Procedural Knowledge					0.88					0.86	
Procedure					0.61					0.59	
Radical Constructivism	0.74					0.78					
Reflection		0.35					0.38				
Reflective Abstraction			0.92					0.91			
Reorganization Hypothesis	0.78					0.7					-0.39
Right							0.3				
Scaff		0.32					0.35				
Scheme	0.71					0.65					
Scheme Theory	0.85					0.85					
Self-Concept											0.6
Should									0.30		
Social Constructivism		0.5					0.45				
Stage			0.44					0.46			
Vygotskian		0.43					0.43				
Wrong				0.36					0.37		