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Abstract

Acquaintance with various ways of inculcating concepts in any studied area of knowledge is one of teachers' duties, particularly mathematics teachers. Studies indicate errors and difficulties when inculcating concepts in mathematics and learning them. Many concepts have different meanings in different contexts. Hence, teachers should deal with the image of the concept by paying attention to its essence. They should also deal with misconceptions developed while inculcating the concept. The present paper describes an activity conducted with 28 teachers studying for an M.Ed. degree in mathematics education at a teacher education college. They attend a course of solid geometry, a topic which is perceived as extremely difficult for the learners. The activity focuses on two ways of inculcating a solid geometry concept: "an angle between a lateral face and a base of a pyramid" as one of the ways for developing spatial orientation. Upon completion of the activity, the paper illustrates the participants' responses regarding their perception of the advantages of each of the ways they experienced and their recommendations as to their preferred ways for inculcating new geometric concepts.

Key words: Different ways of inculcating concepts, From experience to definition, From definition to experience, An angle between a lateral face and a base of a pyramid.

Introduction

The process of inculcating concepts starts at infancy and continues throughout our life. Learners engage in mathematical concepts from kindergarten until they graduate high school. Many concepts, particularly geometric concepts, have visual elements which strongly impact learners (Hershkowitz, 1987). Beyond the formal definition of the concept, learners form an image of it (Tall & Vinner, 1981; Vinner, 1991; 1993). The concept image is a non-verbal representation associated with the concept name in the learners' consciousness. Vinner & Hershkowitz (1983) argue that in order to use the concept in everyday life, learners need a concept image; however, they do not necessarily have to know and master the concept definition. In the thinking process, there is usually an acquaintance with the concept image and not its definition whereas in formal learning the image definition occupies a central place. Vinner & Hershkowitz (1983) maintain that many pupils tend to define geometric concepts in a naïve and simple manner rather than by way of mastery while using the formal definition.

A concept image might sometimes contradict the precise mathematical definition of the concept. When there is a conflict between the formal-mathematical truth and the spontaneous-intuitive feeling, the intuitive feeling usually prevails. Several studies (Hershkowitz & Vinner, 1984; Wilson, 1990) document the gap between a mathematical concept definition and the pupils' use of these concepts. Other studies (Vinner, 1990; Fischbein, 1993; 1996) attempt to explain this issue. Vinner & Hershkowitz (1983) stipulate that there are different theories of learning which claim that definitions are meant to be useful in the process of concept building. They add, however, that empirical evidences show that when an erroneous perception is rooted, pupils ignore the definitions, although they know them by heart. According to Vinner & Hershkowitz (1983), this situation renders teaching-learning of the concept more difficult. Moreover, they argue that teachers, like pupils, build for themselves an image for the concepts they are teaching and this image affects also their teaching process. One should bear in mind that geometric concepts are abstract concepts completely controlled by their definition. That is, all the necessary and sufficient conditions of a concept are included in the definition. Geometric concepts have specific definitions and pupils must learn not only their definitions but understand the importance of the definition as a tool uniting all the necessary and sufficient conditions of the concept. The definition determines the concept scope, namely the category which includes all the examples belonging to it and the way it relates to

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other concepts. Vinner (1982) and Fischbein (1996) argue that the role of definition in geometry is to constitute a final criterion for determining whether a certain example belongs to the concept or not. Moreover, the definition was found to serve as a tool for creating a new concept.

Skemp (1986) discusses experiences with examples of a concept which are required for the creation of that concept. According to Vinner (1991, 1993), learning from examples is more meaningful for learning concepts than learning concepts by means of a definition or by learning the list of general attributes typical of those concepts. Some of the concept attributes are critical to the existence of that concept and they must apply to any form or representation of that concept. Yet, another group of attributes, called non-critical attributes, do not have to apply to every example of the concept. Thus, it would be wrong to associate them as a critical attribute for the description of that learnt concept (Hershkowitz, 1990). The concept definition must encompass the range of all the examples belonging to that concept and only to it. Consequently, when choosing effective ways for facilitating the comprehension of the learnt concept, they should not lead to misconceptions or errors associated with the acquired concept.

The present paper describes two ways for inculcating the concept "an angle between a lateral face and a base of a pyramid". The first way is "from experience to definition" and the second is "from definition to experience". Using these ways might develop the ability of geometric thinking and deepen the concept comprehension and, thus, reduce errors and misconceptions associated with that concept.

Theoretical Background

Many concepts in mathematics have different meanings in different contexts in daily life and in the mathematics language (Patkin, 2011). For example, a solid in daily life means having coherence of particles (as opposed to fluid or gas). In mathematics the meaning is a three-dimensional geometric figure , e.g. a polyhedron or cone. The word side in daily life means any party, team, interest or opinion opposed to another. In the mathematics language it means one of the lines joining the vertices of a polygon. . There are also words that have different meanings in the mathematics language. For example the word base means in daily life a home or an army camp. In mathematics language it has three different meanings. 1. (of a number system). The number represented by the numeral '10' in a positional number system. 2. (of logarithms). The number which, raised to the power of a given logarithm, produces a given number. 3. A line or plane in a geometric figure relative to which the altitude of the figure is measured.

A series of studies which explored learning concepts in plane and solid geometry (Hershkowitz, & Vinner, 1983; Ben-Chaim, Lappan, & Houang, 1989; Hershkowitz,1990) found that pupils at elementary and even junior high school encounter difficulties in the visualization and shift from 2-dimensional to 3-dimensional world and vice versa. For example, when they see a ball in reality, it will look like a circle in a drawing since it is hard to describe 3-dimensional figures in a drawing which has a 2-dimensional figure Grownups, too, find this difficult (Barkai & Patkin, 2012).Furthermore, it is difficult to describe and transcribe that figure. Describing a 3-dimentional figure based on observing it in space is also not an easy task. The process of acquiring the concepts comprises visual elements which affect learners. Some of the concept attributes are critical to the existence of that concept and they must exist in any form or representation of that concept. Yet, another group of attributes, called non-critical attributes, do not have to exist in every example of the concept. Thus, it would be wrong to associate them as a critical attribute for the description of that learnt concept (Hershkowitz, 1990). In mathematics, every concept which is not a primary concept, has a definition encompassing the characteristic attributes of that concept. Moreover, the definition determines the range of all the examples belonging to that concept and only to it. Thus, when inculcating new concepts, we should follow two principles:

1. We should unequivocally distinguish between examples belonging to the definition and examples which do not belong to the definition. For instance: when a prism is defined, a cube is one example of presenting the concept prism because it satisfies all the requirements necessary for defining a prism. A polyhedron like a pyramid is a good example of a solid which does not comply with the definition of prism because not all the requirements necessary for defining a prism are fulfilled.
2. We have to describe only the unique attributes required for defining a certain concept and only them. For example: a prism is a polyhedron built of two congruent faces on two parallel planes. These faces are called bases of the prism and the lateral surface consists of parallelograms (or rectangulars). It is not necessary to talk also about non-critical attributes, e.g. number of the edges of the bases (the bases can be any polygon and not necessarily triangles or quadrangles.

Skemp (1986) stipulates that if learners can categorize new data through their similarity to examples of the concept, it means that they have acquired the concept. He describes the concept acquisition in two stages. One is the first-order abstraction stage, whereby learners create abstraction of the concept attributes based on several experiences. Every additional experience following the first abstraction is examined in relation to the concept and then is defined. Comparison of the additional experiences which follow the abstraction, presents their similarity to the concept and their uniqueness in relation to experiences prior to the concept abstraction. The other is a second-order abstraction stage whereby a group of concepts is generalized into a new concept.

Hershkowitz (1989a) specifies additional criteria which indicate concept acquisition by learners:

1. The ability to explain why a certain object or a certain event are an example or a non-example based on critical or non-critical attributes.
2. The ability to define a concept on the basis of its examples or, at least, using the definition as a criterion for categorization.
3. The ability to use a concept for solving problems, explaining other concepts and explaining the relations between the concept and other concepts.

According to Skemp (1986)] and Vygotsky (1962), pupils sometimes find it hard to distinguish between the concept and its name but the distinction between them is important and critical. A concept is an idea and its name is a tone or sign connected to it. Giving a name to an idea plays an important role in the creation of a new concept. However, the process cannot be shortened by a verbal definition and, in addition, a scientific definition of the concept is usually ineffective. We have to collect examples and experience and only then add the word. The definition is required for further accuracy.

Feldman & Klausmeier (1975) argue that a concept can be learnt on four levels: the concrete level, the identification level, the classification level and the formal level. In the teaching of mathematics the two higher levels – classification and formal – are of interest. Feldman & Klausmeier maintain that it is advantageous to acquire a concept by combining the concept definition and collecting several groups of examples and non-examples of the concept, each differing from the other by one relevant attribute (informed collections). Moreover, Feldman & Klausmeier (1975) claim that a definition by itself is insufficient. A mere definition enables learners to remember a chain of verbal associations only and examples should be attached to it so that they acquire the concept. Similarly in her study of learners' difficulties to cope with a definition, Wilson (1990) claims that turning words into concepts is difficult when no examples are given.

In order to prevent a situation whereby some examples, which pupils learn, serve as a sole framework of reference and visual judgment, we should try judging figures according to their critical attributes beyond visual view. For example, by trial and error of examples and non-examples so that pupils learn about the critical attributes.

About prototype examples and other examples

According to Vinner & Hershkowitz (1983), every concept has examples which are more common and popular and they are called "prototypes". Prototypes are a sub-group of the concept example group and they represent the entire category. A prototype is usually an example with numerous attributes, that is, in addition to the critical attributes of the concept, the prototype has many additional non-critical attributes. When learners relate to a prototype of a concept, they refer to the sub-category of the concept. In their eyes, though, this sub-category becomes the category of the entire concept. Thus, when they are required to examine the category attributes, they examine in fact the attributes of the sub-category. Analysis of a concept means analyzing the concept and exploring its basic critical attributes. These attributes must be included in the object which constitutes an example of the concept while non-critical attributes can change from one example to another (Hershkowitz, 1989a).

In their study, Vinner & Hershkowitz (1983) found several ways of categorizing figures into concepts:

1. Visual category – whereby learners rely on the prototype as a framework of reference, attempting to adjust the figure to the prototype from a visual point of view.
2. Prototype attributes-based category – whereby learners impose the prototype attributes on the example they examine. If some attributes of the prototype do not belong to the examined attribute, pupils exclude the example from belonging to the concept.
3. Analytical category – based on critical attributes. This category is grounded on the critical attributes of a concept as they appear in its definition.

According to Vinner (1983), concept acquisition as manifested by the ability to use it, depends on the creation of a concept image (a group of attributes and images associated with the concept) in people's consciousness. Concept image might comply with the concept definition entirely or partly; it might even contradict it. The role of definition in creating a concept image by the learners is not entirely clear: the concept definition or name might create in the learners' consciousness a whole image of a concept or part thereof or, alternately, create nothing.

Hershkowitz (1990) and Gal (2005) specify that during their lessons in class, many teachers insufficiently use non-examples while inculcating new concepts. Gal (2005) argues that those engaged in developing materials as well as teachers tend to choose certain examples and do not expose the learners to a wide variety of examples. Findings of a study conducted by Hershkowitz & Vinner (1984) illustrated that 5^{th}-8^{th} grades teachers and pre-service teachers learning at teacher education colleges found it difficult to comprehend geometric concepts to the same extent as their pupils. A further study (Hershkowitz, 1987) explored the way of perceiving simple geometric concepts by 5^{th}-8^{th} pupils, pre-service teachers learning in teacher education colleges and elementary school teachers. The findings showed that both teachers and pupils made similar definitions of geometric figures. Furthermore, Hershkowitz (1989a) claims that learners should develop an analytical ability and ground their judgment on the critical attributes of the concept so that they can overcome the erroneous concepts and misconceptions resulting from visual thinking only. She stipulates that learners tend to use a prototype example in order to judge other examples instead of using the concept definition and attributes.

It is therefore important and vital that in-service and pre-service teachers are aware of the processes associated with a concept acquisition and distinguish between the concept image and its correct perception. Moreover, they should be aware of typical mistakes demonstrated during the learning, leading to misconceptions. By means of cognitive conflicts, teachers have to try using these mistakes as leverage for reducing those misconceptions. Being aware of the fact that concepts are acquired in a gradual way, stemming from the development stages and the proper inculcation of concepts, might facilitate and improve the learning process (Patkin, 2010).

Different ways of inculcating concepts – deductive and inductive learning of concepts

The constructivist approach advocates that deep approach is learning whereby new knowledge is connected to previous relevant knowledge which already exists in learners' cognitive structure and is assimilated in it. The interaction between the new and previous knowledge entails a change in learners' cognitive structure. The knowledge increases, develops and is attributed new meanings. Learners are responsible, to great extent, for the process of deep approach. They should clarify to themselves what they already know and in what way they connect what they already know to the new material they are learning now (Ausubel, 1968). The theory conceived by Ausubel attributes special importance to what learners already know. Failure to relate to previous knowledge and rely on it is a waste of expensive intellectual resource as well as of time.

Ausubel describes two directions in the development of a concept system:

1. Progressive differentiation – every concept undergoes a process of additional specification in which its meaning is further expanded and enriched by additional details and examples.
2. Integrative reconcilitation – in this process learners comprehend the wider context of the concept. At the end of the process, the concepts are joined together as principles and theories so that the conflicts between the new and existing concepts are settled, resulting in the generalised completion.

The two processes illustrate the two directions in which concepts develop in deep approach. On the one hand pupils learn more details on low generalization levels, while on the other they obtain wide comprehension of concepts on a high generalization level. Furthermore, Ausubel (1968) claims that organizing concepts in the learners' cognitive structure is done hierarchically, so that some concepts are organized under a more comprehensive concept. We just have to indicate a comprehensive concept and those experiencing deep approach will clearly know what are the additional concepts associated with it.

The deductive approach is the traditional approach to geometry teaching. It is grounded on the notion that structured presentation of contents entails pupils' optimal learning. Teachers who adhere to the deductive approach teach the topics by presenting the basic principles and the definitions thereof. Pupils are required to implement principles by means of examples (Prince & Felder (2006). Hence, deep approach is facilitated by presenting a topic from the general to the examples and the details in a deductive way and it is recommended organizing the lesson in this manner. At the end of the process, teachers examine the pupils' implementation and memorization capabilities. The deductive approach is less constructivist than the inductive approach.

The inductive approach advocates active learning whereby pupils encounter numerous examples and non-examples, experience by themselves and use them in order to learn about generalizations. Teachers' role is to offer opportunities and contexts which will assist learners to attain generalizations and guide them in their work, if necessary. According to Prince & Felder (2006), this approach is more in line with the constructivist approach, based on the requirement that pupils enquire and learn through trial and error. This activity leads to deep approach rather than surface approach which, they argue, transpires in deductive learning. Pupils should be given many non-typical examples which represented the new acquired concepts. Such an exposure will inform pupils of a wider and fuller range of examples, preventing them from concluding that concepts are characterized only by typical examples. Except for the concept examples, non-examples too can facilitate abstraction. Tall (1991) formulates the principle of containing the common attribute in the following way: if all the examples given in a certain context have a certain attribute, then without a contradictory example pupils might assume that the attribute exists also in other contexts.

According to this argument, the two approaches are valuable, each having its own advantages. Felder (1993) claims that some pupils learn better by using the inductive approach whereas for others the deductive approach is better. Those learning according to the inductive approach prefer items of examples and non-examples, aiming to attain generalization. Unlike them, those learning according to the deductive approach prefer getting the definition of the basic principles in order to derive conclusions and examples from them. Disadvantages of the inductive approach reside in the long time required for learning a topic and in the possibility that learners reach a generalization which was not intended by the teachers. Furthermore, there are certain topics which cannot be easily investigated by means of the inductive approach. For example it is not easy to investigate by the inductive approach the concepts of an integral or a differential

Others claim that none of the teaching-learning methods is appropriate for all pupil types. Both approaches are valuable (Patkin, 1996), each having its advantages and integrating both approaches is recommended.
Vygotsky (1962) argues that, on the one hand, concept acquisition depends on concepts developing throughout children's own experience and, on the other, an interaction between this concept system and the theoretical concept system presented to children from outside. In other words, scientific concepts are not acquired by mere memorization and are not embedded in our memory. Rather, they are created and figured by learners' intensive cognitive activity and as a result of the dialectics between two directions: top-to-bottom, from theoretical definitions to examples and bottom-to-top, from examples to the theoretical definitions.

The present paper describes an activity I conducted at a teacher education college. In my opinion, this activity enables those engaged in mathematics teaching to explore two ways for inculcating definitions of new concepts. Thus, they can help learners to overcome their difficulties of acquiring basic concepts of solid geometry as well as to experience the two approaches with their advantages and disadvantages. One way is referred to as "from experience to definition" and the other as "from definition to experience". The idea for developing the activity is grounded on the activity of Hershkowitz (1989b) dealing with the topic of "quadrangle" (a "quadrangle" is a triangle and quadrilateral with a common vertex - a concept conceived for an activity with teachers designed for exploring processes of concept acquisition) and of Patkin (1996) dealing with various ways of inculcating solid geometry concepts.

Below are the two ways – approaches:

From experience to definition is a way by means of which pupils experience consequent attempts of examples and non-examples. Then, pupils are required to consolidate and write a verbal definition of the new concept. This way of teaching demands that learners integrate thinking in the learning process, explore the concept and achieve the concept definition by themselves. The "from experience to definition" way of learning is part of the inductive approach as in this way we go from the particular to the general.

From definition to experience is a way whereby the definition is given to the pupils at the beginning of the learning process. Checking whether the learners understand the meaning of the concept is performed immediately after they have received the exact definition, by validating the definition through examples and non-examples. The objective is to diagnose whether the definition has been accurately internalized or some disruptions in the comprehension of the defined new concept have transpired. The "from definition to experience" way of learning is part of the deductive approach as in this way we go from the general to the particular (Prince & Felder, 2006).

As indicated above, there are opponents and proponents for each of the approaches – the inductive and the deductive – while others advocate working with both approaches together.

Short description of the activity performed in the course which I taught

Twenty-eight elementary and junior high school mathematics teachers took part in the activity. All the teachers hold a B.Ed. and M.Ed. degree in mathematics and have more than a 3-year seniority. They attended a semestrial course (15 weekly sessions, each 2-hour long) dealing with solid geometry within the framework of their studies towards an M.Ed. in mathematics education. The participants were divided into two groups. One group comprised 14 teachers (nine elementary school teachers and five junior high school teachers) who studied in the "from experience to definition" way. The second group included the other 14 participants (nine elementary school teachers and five junior high school teachers) who studied in the "from definition to experience" way. The first four lessons (eight academic hours) were dedicated to the theory of geometric thinking development (Van Hiele, & Van Hiele, 1958; Van Hiele, 1987; 1999), and to varied activities in solid geometry, such as: finding the number of cubes which comprise solids; view of solids from different perspectives; drawings of various solids; and providing formal definitions to familiar geometric concepts. For example: a cube is a polyhedron which is a regular square prism whose faces are all of the same area. Another definition for a cube is a regular solid consisting of six congruent squares, etc.

In the fifth lesson the participants were divided into two groups in order to experience the two different ways for acquiring the concept which was unfamiliar to them: "an angle between a lateral face and a base of a pyramid".
It is worthwhile mentioning that according to the Israeli curriculum, the topic of solid geometry has not been thoroughly studied for many years. Consequently, despite their formal education in mathematics, when asked by the lecturer (writer of the paper) prior to the course, the teachers answered they were unfamiliar with this concept.

"An angle between a lateral face and a base of a pyramid" – definition:

- The angle formed between the altitude of the lateral face and the common edge with the basis and the projection of this altitude on the basis plane.
  Or:
- **The angle between a lateral face and the base of a pyramid is the angle formed between the two perpendiculars to** the common edge between the basis and the lateral face which is the intersecting straight line.

A. From experience to definition

The 14 teachers in that sub-group were given the new concept without any explanation or definition: "An angle between a lateral face and a base of a pyramid". In order to learn the concept they were given worksheets. The teachers were required to cover every page, before starting to work, with a paper page, move the paper downwards until the first upper broken line, responding to the question presented next to the uncovered drawing. Then, they had to move the paper to the second line, respond to the next question and so on and so forth with all the 22 items of examples and non-examples. After completing the activity the participants had to define (by themselves based on their experiences) the concept: "An angle between a lateral face and a base of a pyramid". The activity itself is based on examples and non-examples designed to gradually lead the learners to the consolidation of the accurate definition of the new acquired concept. The examples and non-examples were collected from a database of correct and incorrect pupils' answers given in their homework and tests, requiring them to draw an angle between a lateral face and a base of a pyramid.

Instructions: you have to answer each question while hiding the questions following it. Each time uncover only the next question and, in this way, move forward.

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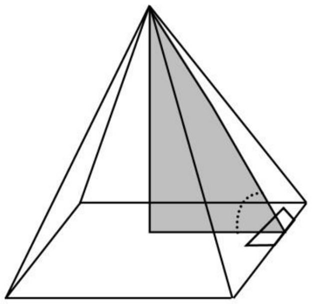
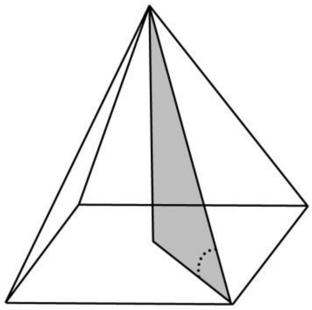
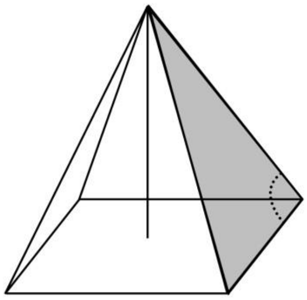
<tr><td></td><td>1. The angle indicated in the drawing is an angle between a lateral face and a base of a pyramid</td></tr>
<tr><td></td><td>2. Is the angle indicated an angle between a lateral face and the basis?

Yes / No</td></tr>
<tr><td></td><td>3. The answer to the previous question is "no"
Is the angle indicated an angle between a lateral face and the basis?
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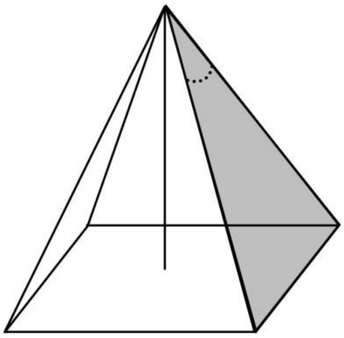
<tr><td></td><td>4. The answer to the previous question is "no"
Is the angle indicated an angle between a lateral face and the basis?
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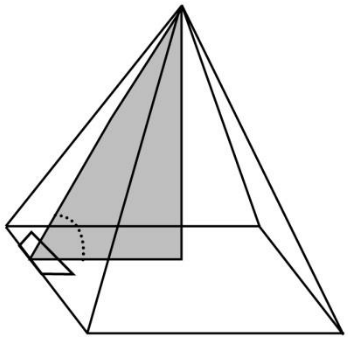
<tr><td></td><td>5. The answer to the previous question is "no"

Is the angle indicated an angle between a lateral face and the basis?
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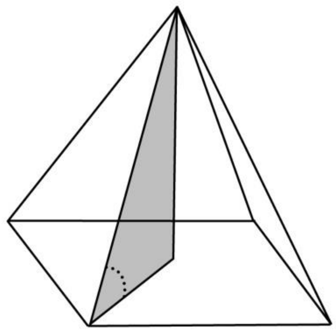
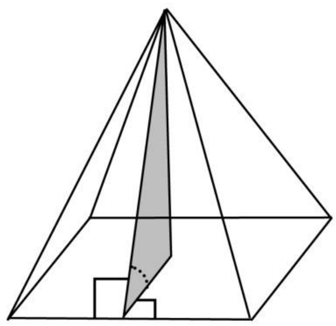
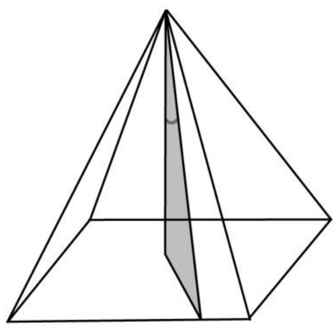
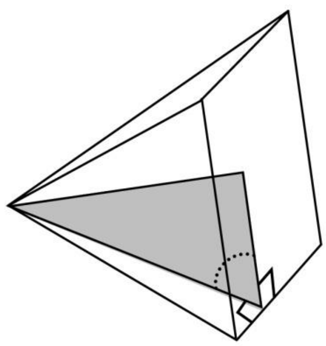
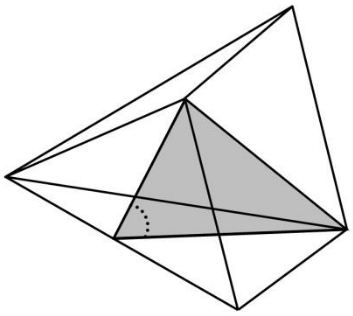
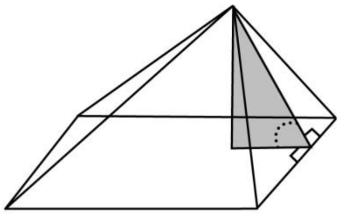
	6. The answer to the previous question is **"yes"** Is the angle indicated an angle between a lateral face and the basis? Yes / No
	7. The answer to the previous question is **"no"** Is the angle indicated an angle between a lateral face and the basis? Yes / No
	8. The answer to the previous question is **"yes"** Is the angle indicated an angle between a lateral face and the basis? Yes / No
	9. The answer to the previous question is **"no"** Is the angle indicated an angle between a lateral face and the basis? Yes / No



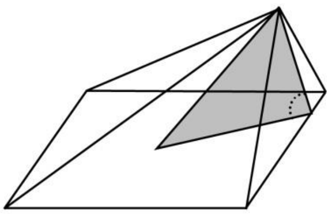
10. The answer to the previous question is **"yes"**

Is the angle indicated an angle between a lateral face and the basis?
Yes / No



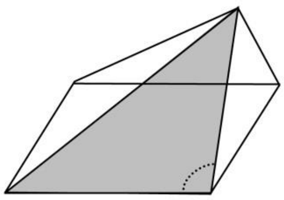
11. The answer to the previous question is **"no"**

Is the angle indicated an angle between a lateral face and the basis?
Yes / No



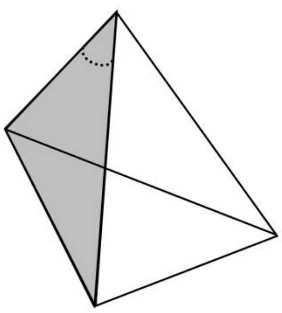
12. The answer to the previous question is **"yes"**

Is the angle indicated an angle between a lateral face and the basis?
Yes / No



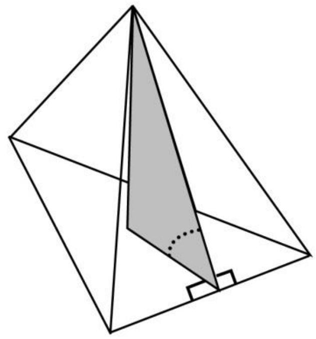
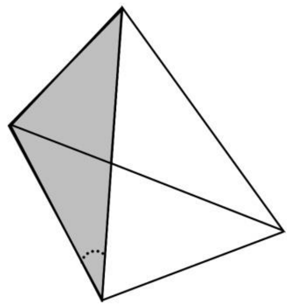
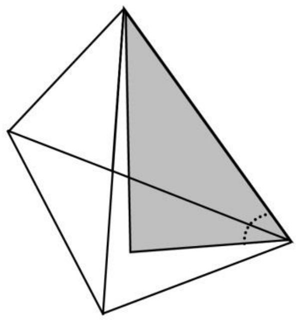
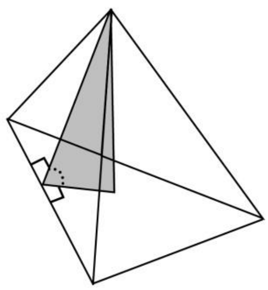
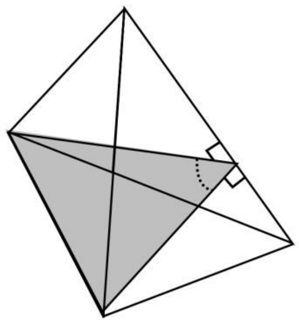
13. The answer to the previous question is **"no"**

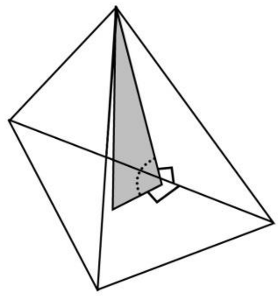
Is the angle indicated an angle between a lateral face and the basis?
Yes / No



14. The answer to the previous question is **"no"**

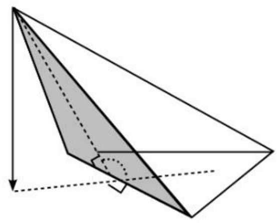
Is the angle indicated an angle between a lateral face and the basis?
Yes / No

	15. The answer to the previous question is **"no"** Is the angle indicated an angle between a lateral face and the basis? Yes / No
	16. The answer to the previous question is **"yes"** Is the angle indicated an angle between a lateral face and the basis? Yes / No
	17. The answer to the previous question is **"no"** Is the angle indicated an angle between a lateral face and the basis? Yes / No
	18. The answer to the previous question is **"no"** Is the angle indicated an angle between a lateral face and the basis? Yes / No
	19. The answer to the previous question is **"yes"** Is the angle indicated an angle between a lateral face and the basis? Yes / No



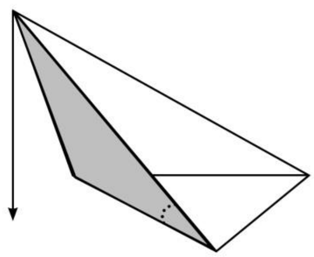
20. The answer to the previous question is **"no"**

Is the angle indicated an angle between a lateral face and the basis?
Yes / No



21. The answer to the previous question is **"yes"**

Is the angle indicated an angle between a lateral face and the basis?
Yes / No



22. The answer to the previous question is **"yes"**

Is the angle indicated an angle between a lateral face and the basis?
Yes / No

The answer to the previous question is **"no"**

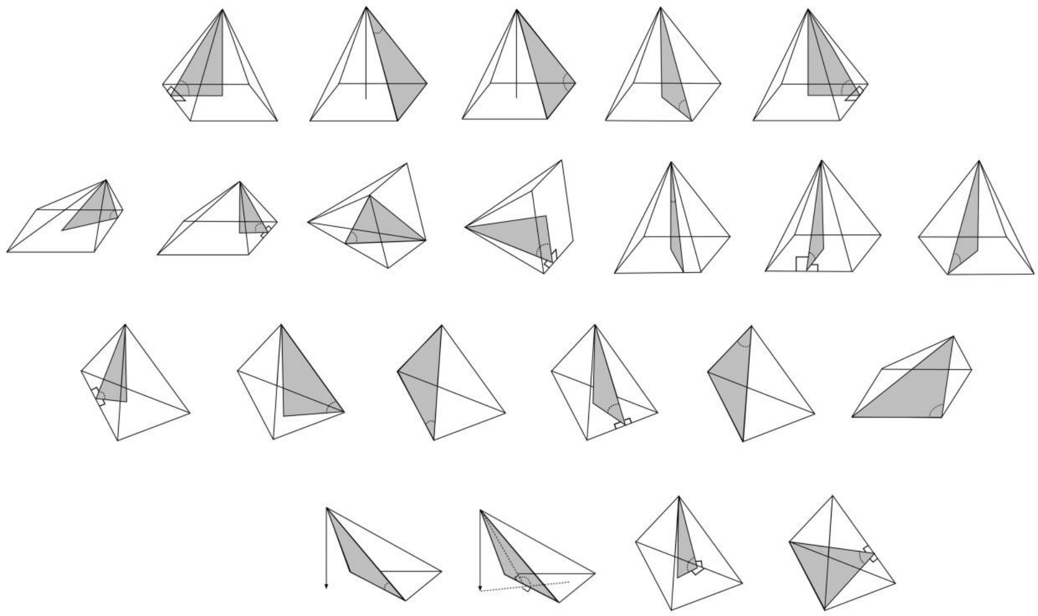
And now
Based on all the drawings and answers you obtained, please define:

An angle between a lateral face and a base of a pyramid is:____________________________
____

B. From definition to experience

The other 14 teachers who constituted the second group received a definition of the concept: an angle between a lateral face and a base of a pyramid. They were shown 22 drawings of pyramids within which an angle was indicated. The drawing where identical to those given in way A - 'from experience to definition'. The teachers were asked to mark those pyramid drawings whose angle complies with the definition "an angle between a lateral face and a base of a pyramid" given to them.
Definition: an angle between a lateral face and a base of a pyramid is the angle formed between the altitude of the lateral face and the common edge with the basis and the projection of this altitude on the basis plane. That is, the angle formed between the two perpendiculars to the common edge between the basis and the lateral face.

Instructions: you have to identify among all the indicated angles the **angle between a lateral face and a base of a pyramid.**



It is important to point out that an additional version with a higher difficulty level for the inculcation according to the "from definition to experience" approach could be asking the learners to draw various examples corresponding to the concept "an angle between a lateral face and a base of a pyramid".

In the sixth lesson the participants were required to experience also the other inculcation way. As learners, those who worked in the previous lesson in the "from experience to definition" way used now the "from definition to experience" way whereas those who worked first with the "from definition to experience" way later implemented the "from experience to definition" way.

After completing the experience a discussion was conducted, aiming to develop mathematical discourse about learners' typical mistakes in perceiving the learnt concepts. Within this framework, they analyzed the mistakes found, while relating to the reasons leading to those mistakes.

From the seventh lesson and until the end of the semester the teachers were asked to try implementing the two ways in their class at school and to report at the end of the semester which teaching was better for them and led to a good inculcation of the concepts they taught their pupils. They were required to present an example whereby they inculcated concepts in the two different ways. In order to enable them to fully implement these ways, each teacher chose concepts which matched the curriculum in his or her class. That is, the concepts did not have to be from the field of solid geometry but from any topic of geometry which suited them. The last lesson was dedicated to discussion and summary. The teachers indicated what, in their opinion, were the advantages of each way, pointing out a preferred way, if they had one, for learning new concepts and the teaching thereof.

Below are some of the phrases said during that session, which illustrate the advantages and disadvantages of each way and present material for thought for teacher educators in this field.

"From experience to definition"
Advantages: *"this is the natural way by which children learn and acquire concepts. For example, how do we explain to children that touching hot water might cause burns? If the child touches once very hot water and the touch is painful, he or she will understand that this is not recommended. Similarly, many grownups tend to acquire concepts in this way"* (A.G. participant in the activity).

One of the participants formulated it well: *"Learning according to the "from experience to definition" way leads to deep approach and to a learning process encompassing internalization of information in a way which allows generalization and conclusion drawing as well as an indirect change of behavioral patterns by creating and reinforcing pupils' motivation as a result of being put at the center of practice. Pupils take part in shaping the concept and understanding it and, hence, their mastery of the learnt material is enhanced"* (K.R. participant in the activity).

Disadvantages: learning in this way is sometimes based on partial learning. *"A partial collection of examples and non-examples, undermining the consolidation of the concept formal definition"* (D.H. participant in the activity).

"From definition to experience"
Advantages: *"When we start by giving a formal definition, the definition constitutes a distinguishing factor between examples which represent the concept and non-examples which accompany it"* (R.M. participant in the activity).

Disadvantage: *"The effect of the intuitive image is sometimes stronger than the formal definition of the learnt concept"* (A.D. participant in the activity).

Moreover, the participants presented several questions and queries:

- Can we conclude from a single activity about the effectiveness of one teaching way over another for a long period of time?
- Is it recommended adopting only one way or using various ways, according to learners' learning style as well as their ability to cope with formal definitions?
- Is it possible and even recommended inculcating concepts in a way which integrates the two approaches? That is, we can start teaching by experience and, later, ask the learners to define the learnt concept in a formal way. Then we can discuss the definition, ending the process by "validating" the acquired concept by means of additional examples and non-examples. Or we can start the process in the opposite way.
- Is a concept definition achieved only by examples and non-examples (i.e. inductive definition) sufficient and does not lead to erroneous or partial concepts? Based on the fact that in some cases the number of examples and non-examples is infinite, can an inductive definition unequivocally "cover" the defined concept?

Discussion

In the process of concept acquisition it is critical for each concept to be inculcated so that it is most meaningful for the learners rather than a mere collection of meaningless words. While conceptualizing concepts of plane and solid geometry one can notice stages which are in line with the Van-Hiele's theory of geometric thinking development (Patkin, 2010). At first learners are impressed by the figure in its entirety and, as they advance, they relate also to the critical and non-critical attributes of that figure. Hence, we the teachers, should know and master various ways of concept inculcation, design various strategies for teaching the concepts and integrate activities based on analytical relation between the concept critical attributes by series of examples and non-examples.

When presenting the activity described in this paper, I do not favor one way more than the other. Teachers should be aware of the two approaches or the integration thereof while teaching. The very increased awareness of existing alternatives for inculcating a concept improves, in my opinion, our teaching way. Based on my personal experience as a mathematics teacher educator I can attest that I have used these ways not only in the teaching of geometrical concepts but also in other topics of mathematics. As a lecturer, collecting examples and non-examples assists me to expose learners to existing common errors and emerging misconceptions. Consequently I recommend applying these ways particularly with regard to concepts which are "difficult" to teach and learn, aiming to improve educators' teaching methods and make them raise questions related to effective teaching methods. All these might improve pupils' learning ways and satisfy their needs.

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