Taxonomy of Key Terms for Mathematics Education

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Taxonomy of Key Terms for Mathematics Education

Paola Castro, Pedro Gómez

Abstract

We present the process of developing a taxonomy of key terms for Mathematics Education. We build on the existing taxonomy of key terms that has been used in an open access document repository. Additionally, we took into account terms that have been established in encyclopedias of the discipline and the frequency of use of keywords in specialized journals that were indexed in Scopus and Web of Science. We made a review of synonymy between these terms and the terms of the existing taxonomy. We included in our proposal the terms that are relevant given their frequency of use in the journals. We removed from the existing taxonomy the terms that are little used in practice. The new taxonomy is organized in six main categories: approach, educational level, foundations of Mathematics Education, research in Mathematics Education, pedagogical notions and mathematical content. This proposal was validated in three phases by researchers, innovators in Mathematics Education, and editors of specialized journals and experts who lead associations and events in the discipline.

Introduction

The specific descriptors of a discipline are tools that allow the knowledge emerging from it to be synthesized with key concepts. In general, the Mathematics Education community continues to assign in an open way the key terms of its documents in journals and events of the discipline. This highlights the need to generate a controlled vocabulary that facilitates the characterization of documentation.

Currently, Mathematics Education is treated as a key term in thesauri and general databases (ERIC, 2017; UNESCO, 2018). Additionally, there is a classification of Mathematics Education and Computer Science descriptors whose use generates difficulty in identifying terms—MathEduc (FIZ Karlsruhe, 2010, 2019). This classification of subjects, besides not being exclusive to Mathematics Education, is not easy to use for the classification of works in the discipline. Nor is it characterized by its practicality in the search for documents that deal with specific aspects.

Some researchers in Mathematics Education recognize the lack of a closed list of descriptors that is specific to the discipline, that allows the identification of the particular characteristics that distinguish it from other disciplines and that makes it possible to recover information produced in it (Adamuz-Povedano, Jiménez-Fanjul,
Another approach to the problem of establishing a controlled vocabulary for Mathematics Education is given in the context of an open access repository (http://funes.uniandes.edu.co), in which a specific taxonomy is used to code and organize documents (Gómez & Cañadas, 2013). However, its relationship with a concrete conceptual approach and its use in a particular context makes the taxonomy susceptible to be evaluated and adjusted according to the current state of knowledge in the discipline, at a global level. Based on this panorama, we provide a specific taxonomy of Mathematics Education that reflects the focuses of interest of the international community, which is endorsed by it, and which allows the characterization and organization of the knowledge that is produced in the discipline. Firstly, we describe the approaches that have been made to establish and classify the discipline’s own descriptors. Then, we present the conceptual framework on which our proposal is based, specify our aim, detail the method used and provide the structure of the new taxonomy.

Descriptors of Mathematics Education

With regard to Mathematics Education, several approaches have been made to the classification of key terms specific to the discipline. According to their level of specificity, we distinguish the descriptors as general and specific, and describe them below.

General Descriptors

As general descriptors, we find that, in the UNESCO Thesaurus (UNESCO, 2018), Mathematics Education is associated with two terms: learning statistics (as a specific concept, indicating the more restricted term) and arithmetic knowledge (as a related concept, indicating the relationship between descriptors united by an association of ideas). In the ERIC database (ERIC, 2017), Mathematics Education falls under the category of Mathematics. The term is related to Education. Associated terms include College Mathematics, Elementary School Mathematics, Mathematics Activities, Mathematics Instruction, High School Mathematics, and STEM Education. In Mathematics Education, Bloom’s revised taxonomy has been used to analyze teaching, learning and assessment. A variant of this proposal has been used to interpret a mathematical concept in order to establish descriptive terms (Long & Dunne, 2014). However, this taxonomy is too general, as it was not developed to address the discipline specifically (Radmehr & Drake, 2019).

Specific Descriptors

As approaches to specific descriptors of Mathematics Education, we identified the classification of subjects used in the MathEduc database, formerly MathDi (FIZ Karlsruhe, 2010, 2019), the production of descriptors for
document searches in Scopus and Web of Science (Adamuz-Povedano et al., 2013; Jiménez et al., 2011), and the development of a taxonomy of key terms in Mathematics Education (Gómez & Cañadas, 2013).

**MathEduc Database**

MathEduc proposes a classification of Mathematics Education topics. These topics are organized into 16 main areas that are related to both Mathematics Education and computer science. Within these areas, there are specific themes that are associated with the level of education or type of training. For all the mathematical contents, the first specific subject is the same: “comprehensive works on (...) and the teaching of (...)”. The other themes are usually more specific in terms of content. In some cases, the description of the topics includes a reference to other categories. As an example, we present, in Figure 1, a section of the category corresponding to arithmetic, number theory and quantity. We use red boxes to represent the main area and topic, blue to highlight the level of education and green for the topics covered in other areas.

```
<table>
<thead>
<tr>
<th>F</th>
<th>Arithmetic. Number theory. Quantities</th>
</tr>
</thead>
<tbody>
<tr>
<td>F10</td>
<td>Comprehensive works on arithmetic and the teaching of arithmetic</td>
</tr>
<tr>
<td>F20</td>
<td>Prenumerical stage. Number concept, counting</td>
</tr>
<tr>
<td>F30</td>
<td>Natural numbers and operations on natural numbers. Place value. Pencil and paper arithmetic, mental arithmetic</td>
</tr>
</tbody>
</table>
```

**Figure 1. Arithmetic. Number theory. Quantities Category**

The subject classification used in the MathEduc database does not make it easy to classify and search for documents. In the case of the arithmetic category, number and quantity theory (Figure 1), a specific topic includes several issues that, although related, differ in their meaning (e.g., operations in natural numbers and positional value in F30). In addition, other issues have to be identified in other categories (e.g., estimates in N20).

**Descriptors for Document Search**

Some researchers in Mathematics Education recognize the lack of a closed list of descriptors that is specific to the discipline, that allows the identification of the particular characteristics that distinguish it from other disciplines and that makes possible the recovery of information produced in it. The works of Jiménez et al. (2011) and Adamuz-Povedano et al. (2013) make an approach to the determination of basic descriptors that characterize the scientific production of the Mathematics Education community that is indexed in the Scopus and Web of Science databases. The purpose of these works is to present a list of descriptors that can be used in the search for documents of the discipline in the previously mentioned databases.

In the first proposal, the authors establish three groups of categories: descriptors of Mathematics, descriptors of Education and descriptors of Mathematics Education. The second proposal coincides with the first in initially
organizing the descriptors by categories (specific descriptors of Mathematics and specific descriptors of Education) to then provide a unified list that characterizes Mathematics Education in the databases. Although both works highlight the need to generate a vocabulary that allows the characterization of documentary production in Mathematics Education, they provide descriptors that are too general to account for the phenomena and problems that are dealt with in a document. For example, the use of a term such as learning does not make it possible to establish which aspects of this subject is being dealt with (learning theories, learning expectations, difficulties or errors?).

**Taxonomy of Key Terms in Mathematics Education**

Gómez and Cañadas (2013) provide a classification and hierarchy of descriptors specific to the discipline. In their proposal, they organize the key terms into purpose, educational level and subject. The purpose characterizes the type, intent and usefulness of the document, and the educational level refers to the level of training of the subjects referred to. With regard to the subject, the authors differentiate the terms related to Mathematics Education from those that deal with mathematical contents; these, in turn, are divided into school mathematics and higher mathematics contents. The terms that address specific issues of Mathematics Education arise from a specific curricular approach (Rico, 1997). From this approach, the authors tackle teaching, learning and assessment, and support the categories of the taxonomy according to four dimensions of the curriculum (conceptual, cognitive, formative and social) at five levels: purposes, disciplines, educational system, teacher planning and local planning.

The existing taxonomy arises from a particular theoretical approach and is designed with a practical purpose. The authors produced a hierarchical structure of key terms that allowed for the systematic classification, in the open access digital repository Funes (http://funes.uniandes.edu.co), of documents produced by the Mathematics Education community. The use of taxonomy in the codification process of the documents hosted in the repository began in October 2009. After 10 years, it is relevant to verify its effectiveness and evaluate if it meets the current interests of researchers and mathematics educators in a global context.

**Conceptual Framework**

Controlled vocabularies are used for the representation of content objects in knowledge organization systems (National Information Standards Organization, 2005). The selection of terms to be included in a controlled vocabulary should be based on three elements: the natural language used to describe content objects, the language of the users, and the needs and priorities of the organization. Vocabulary control is carried out through three methods:

(a) definition of the scope or meaning of the terms,

(b) use of the equivalence relation to link synonymous and quasi-synonymous terms, and

(c) distinction between homograph terms.
The controlled vocabularies focus on content. However, it is possible to address other aspects of the documents such as authorship, location, format, language and place of publication. Controlled vocabularies can be lists of terms, thesauri or taxonomies. In the lists of terms there is only one relation of belonging of each term to the list. Thesauri show the various relationships between terms by standardized relationship indicators. Taxonomies present key terms organized hierarchically into categories and subcategories.

A taxonomy is defined as a structure that organizes knowledge according to the hierarchy of concepts that underlie it (Paukkeri, García-Plaza, Fresno, Unanue, & Honkela, 2012). Among the applications that have been recognized for taxonomies, we highlight their use in information management, in the organization and categorization of data, and in the search for content (Sujatha, Bandaru, & Rao, 2011). A taxonomy allows the organization of content based on the standardization of its descriptors, provided that it has a defined content and its related metadata (Engel, Pryde, & Sappington, 2010).

Taxonomy is consolidated as an effective means the management and access to digital information. The method for generating a taxonomy is associated with several factors such as the nature of the data, the semantic implication and the type of application it will have (Irfan, Khan, Abbas, & Shah, 2019). In this sense, it is possible to organize short data, such as tags or keywords, whose nature is concise and represent them in a way that easily identifies the hierarchical relations between them. From a semantic approach, the extraction of concepts that are considered relevant in the knowledge that is covered by the taxonomy is used.

Irfan et al. (2019) recognize that the incorporation of semantics in the generation of a taxonomy by existing computational techniques is complex due to the differences that can be established in the meaning of the terms or concepts that make up the taxonomy. In addition, the automatic generation of labels by these methods is often not very precise and significant compared to the manual assignment of terms. Hence, the importance of having a conceptual basis to define and organize terms. In fact, the literature recognizes that very few automatic techniques have addressed the problem of determining hierarchy, since the hierarchical structure must reflect the essence of the relationships between terms.

Aim

Our aim is to produce a taxonomy of key terms that emerge from the knowledge produced by the international Mathematics Education community. To characterize the community’s knowledge in a hierarchy of key terms, we combine the use of the existing taxonomy in the coding of documents in a digital repository of open access documents in Mathematics Education and the most used keywords in the Mathematics Education journals that were indexed in Scopus and Web of Science in 2017. The new taxonomy has been endorsed by the community and allows to link the current state of knowledge, according to publications in research journals specialized in the discipline, with the practical use of the existing taxonomy (Gómez & Cañadas, 2013). This relationship is shown in Figure 2.
Method

We carried out a systematic process to produce a taxonomy of key terms in Mathematics Education. We identified the terms that are relevant to the discipline due to their frequency of use in publications of the discipline at an international level. We generated the hierarchical list of terms and carried out a process of validation of the taxonomy in relation to its structure and the labels used. In the following, we describe the sources of information and procedures.

Sources of Information

We used the same type of information sources that have been used in the production of taxonomies for other fields (Aadland & Aaboen, 2020; Fellnhofer, 2019; Klassen & Donald, 2020; Pertegal-Vega, Oliva-Delgado, & Rodríguez-Meirinhos, 2019). We used four sources of information for the production of the new taxonomy:

- The first is the terms proposed in the encyclopedias published specifically in Mathematics Education (Grinstein & Lipsey, 2001; Lerman, 2014).
- The second is the list of key terms from articles published between 2007 and 2017 in Mathematics Education journals which, in 2017, satisfied the condition of being indexed in the Scopus and Web of Science databases.
- The third is a discipline-specific taxonomy of key terms (Gómez & Cañadas, 2013).
- The fourth corresponds to the frequency of use of the key terms that are used in the open access digital repository in Mathematics Education (Funes) to encode the documents hosted in it.

Below, we describe each source of information.
Terms Defined in Specialized Encyclopedias

We reviewed the 450 terms in the first specialized encyclopedia (Grinstein & Lipsey, 2001). Although aspects such as curriculum and assessment are included, this encyclopedia emphasizes the mathematical content that is taught at different educational levels.

We analyzed the 162 defined terms, and their related keywords, in the second encyclopedia (Lerman, 2014). In this publication, the inclusion of theoretical proposals that are specific to Mathematics Education that address the problem of teaching and learning mathematics is evident.

Key Terms for Articles in Indexed Journals

We identified the specialized journals in the discipline that were indexed in the Scopus and Web of Science databases in 2017. We took these databases because of their international recognition in relation to the visibility of knowledge arising from research. Additionally, we focused on specific journals in Mathematics Education since some studies show that the journals indexed in Scopus with the highest number of articles in the discipline are those specialized in it (Cruz-Ramírez, 2018). In fact, it is evident that, until 2018, of the 15 journals with the highest number of articles, only 2 were not Mathematics Education journals (Cruz-Ramírez & Rodríguez-Devesa, 2019). The percentage of articles published in non-specialized journals, compared to the percentage of articles of the discipline published in specialized journals, is less than 7%.

The following are the journals that, in 2017, satisfied the condition of being indexed in the Scopus and Web of Science databases: Journal for Research in Mathematics Education; International Journal of Science and Mathematics Education; Mathematical Thinking and Learning; Educational Studies in Mathematics (An International Journal); Eurasia Journal of Mathematics, Science and Technology Education; and Revista Latinoamericana de Investigación en Matemática Educativa. Our source of information is the lists of key terms in these publications, which emerge from articles published between 2007 and 2017. In Table 1, we show the starting year of publication of the journals that are indexed in the databases, the number of documents published between 2007 and 2017, and the number of key terms that are obtained from these documents. In the case of the journal Mathematical Thinking and Learning, articles published before 2019 did not include key terms.

<table>
<thead>
<tr>
<th>Journal</th>
<th>Start Year</th>
<th>Articles</th>
<th>Key Terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Journal for Research in Mathematics Education</td>
<td>1996</td>
<td>249</td>
<td>451</td>
</tr>
<tr>
<td>International Journal of Science and Mathematics Education</td>
<td>2003</td>
<td>807</td>
<td>2582</td>
</tr>
<tr>
<td>Educational Studies in Mathematics (An International Journal)</td>
<td>1968</td>
<td>698</td>
<td>2329</td>
</tr>
<tr>
<td>Eurasia Journal of Mathematics, Science and Technology Education</td>
<td>2006</td>
<td>926</td>
<td>2760</td>
</tr>
<tr>
<td>Revista Latinoamericana de Investigación en Matemática Educativa</td>
<td>2009</td>
<td>136</td>
<td>438</td>
</tr>
<tr>
<td>Mathematical Thinking and Learning</td>
<td>2009</td>
<td>125</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>2941</td>
<td>8560</td>
</tr>
</tbody>
</table>
Specific Taxonomy in Mathematics Education

We took, as a basis, the existing taxonomy in Mathematics Education (Gómez & Cañadas, 2013) to produce our proposal. The authors propose a discipline-specific taxonomy based on a solid conceptual framework. The first aspect that these authors address is the key terms called purpose and educational level. The purpose characterizes the type, intent and usefulness of the document. A document can be a research paper, an essay, an innovation paper or an activity paper. The educational level refers to the type of training of the subjects referred to in the document: pre-school education, primary education, secondary education, upper secondary education, adult education, postgraduate education, professional education, undergraduate education, all educational levels, no educational level and other educational level.

The taxonomy differentiates key terms referring to mathematical content from those referring to Mathematics Education. To establish the terms associated with mathematical content, the authors based themselves on the classification used by TIMMS (Mullis et al., 2005) and TEDS-M (Tatto, Schwille, Schmidt, Ingvarson, & Beavis, 2006) and distinguished school mathematics from higher mathematics. Mathematics Education subjects emerge from a curricular approach that addresses four central issues: knowledge to be taught, learning, teaching methods and the assessment (Rico, 1997). The four issues give rise to the conceptual, cognitive, formative and social dimensions, and to five levels (purposes, disciplines, educational system, teacher planning and local planning). This curriculum theory supports nine categories of key terms:

(a) education system,
(b) education center,
(c) classroom,
(d) student,
(e) teacher,
(f) learning,
(g) teaching,
(h) assessment, and
(i) curriculum.

Additionally, the authors include categories associated with other notions of Mathematics Education, research and innovation in Mathematics Education, and Mathematics Education and other disciplines.

For Mathematics Education topics, the taxonomy is made up of 236 key terms that are organized into 12 main categories. There are 89 school mathematical terms (organized into 8 categories) and 16 higher mathematical terms. In the last section, the authors did not establish any hierarchy.

The construction of the taxonomy was based on MathEduc’s classification of topics (FIZ Karlsruhe, 2010), so that every key term in that database would have an equivalent term in the proposal. To ensure relevance to the discipline, Gómez and Cañadas (2013) reviewed the way in which some research journals, conference proceedings and national and international databases assign key terms to their papers, and explored the usefulness of the taxonomy with various experts in the discipline. By doing so, the authors guaranteed the
relevance of the taxonomy and its relationship to the key terms that have been traditionally used in the
discipline.

Key Terms in a Digital Document Repository

Funes, the digital document repository in Mathematics Education, makes available to the community of
mathematics educators the documents that are not restricted by copyright and that can support the work of this
community. Its content is available to the entire public. There are no restrictions on access to the portal and the
documents are not differentiated for access. The documents are classified into different types, according to their
purpose: research, essays, curricular innovations or tasks.

The documents hosted in the repository can be articles, book chapters, theses, reports and presentations of
meetings or working papers. In order for a document to be published in Funes, it must go through a codification
process that establishes its focus and educational level, as well as its key terms in relation to the topics of
curricular theory and the mathematical content it addresses. The assignment of the key terms of each document
is made from the existing taxonomy in Mathematics Education that we described in the previous section
(Gómez & Cañadas, 2013). The hierarchy levels of the taxonomy allow relationships between key terms to be
identified and provide information about the number of documents associated with each key term
(http://funes.uniandes.edu.co/view/subjects/).

We decided to take the frequencies of the key terms of the documents hosted in the Funes repository as a source
of information for the following reasons:

(a) it hosts curricular innovations and essays, not only research documents;
(b) it has diverse sources of open access information (divulgation and research journals, memories of
events, institutional repositories, and authors who share their work autonomously);
(c) on March 6th, 2020, it had more than 12000 documents, and
(d) it is focused on the Ibero-American community, whose documental production is on the rise.

We highlight the last aspect since it has been identified that, in the Scopus database, production is focused on
non-Spanish speaking countries—United States, United Kingdom, Australia, Turkey, Canada and Germany
(Cruz-Ramirez & Rodriguez-Devesa, 2019).

Procedures

The production of the new taxonomy, specific to Mathematics Education, involves five phases: (a) the
identification of key terms of the discipline, (b) the revision of the synonymy of these key terms in relation to
the key terms of the existing taxonomy (Gómez & Cañadas, 2013), (c) the identification of terms to be included
in the new taxonomy, (d) the selection and organization of these terms and (e) their validation. We describe
these phases below. Figure 3 presents the data collection and organization procedures leading to the production
of the new taxonomy.
Identification of Key Terms in the Discipline

We identified the key terms in Mathematics Education in the first two sources of information. To establish which terms are relevant to Mathematics Education, we recognized the theoretical trends of the terms included in the encyclopedias. Regarding the terms assigned in articles published in the journals indexed in Scopus and Web of Science, we unified a list of terms with their respective frequency. For example, we found that the term Decision making has a total frequency of 13 in the journals considered (see Table 2).
Table 2. Frequency of Use of the Term “Decision making” in Journals

<table>
<thead>
<tr>
<th>Journal</th>
<th>Key term</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eurasia Journal of Mathematics, Science and</td>
<td>Decision making</td>
<td>3</td>
</tr>
<tr>
<td>Technology Education</td>
<td></td>
<td></td>
</tr>
<tr>
<td>International Journal of Science and Mathematics</td>
<td>Decision making</td>
<td>1</td>
</tr>
<tr>
<td>Education</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Journal for Research in Mathematics Education</td>
<td>Decision-making</td>
<td>2</td>
</tr>
<tr>
<td>Eurasia Journal of Mathematics, Science and</td>
<td>Decision-making</td>
<td>2</td>
</tr>
<tr>
<td>Technology Education</td>
<td></td>
<td></td>
</tr>
<tr>
<td>International Journal of Science and Mathematics</td>
<td>Decision-making framework</td>
<td>1</td>
</tr>
<tr>
<td>Eurasia Journal of Mathematics, Science and</td>
<td>Multiple criteria decision making (MCDM)</td>
<td>1</td>
</tr>
<tr>
<td>Technology Education</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eurasia Journal of Mathematics, Science and</td>
<td>Multiple criteria decision making (MCDM)</td>
<td>1</td>
</tr>
<tr>
<td>Technology Education</td>
<td>methodology</td>
<td></td>
</tr>
<tr>
<td>International Journal of Science and Mathematics</td>
<td>Teacher decision-making</td>
<td>1</td>
</tr>
<tr>
<td>Education</td>
<td></td>
<td></td>
</tr>
<tr>
<td>International Journal of Science and Mathematics</td>
<td>Teachers’ decision-making</td>
<td>1</td>
</tr>
<tr>
<td>Education</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>13</td>
</tr>
</tbody>
</table>

Revision of the Synonymy of Key Terms

For each key term in each journal, we analyzed its synonymy in relation to the key terms of the existing taxonomy (Gómez & Cañadas, 2013). To do so, we identified four possibilities, so that each term can

(a) be synonymous or identical to a term in the existing taxonomy,

(b) be included in some category of the existing taxonomy,

(c) not be included in the existing taxonomy but be relevant to the discipline (included in encyclopedias) and

(d) not be relevant to the discipline.

As an example, in the analysis of the key term Engineering and mathematics programs (STEM) from the Eurasia Journal of Mathematics, Science and Technology Education, we saw that it is related to the term Relationship of Mathematics Education with other areas of the existing taxonomy. Although the term, as we present it, has frequency 1 in this journal, we decided to mark it as relevant due to the use of the generic term STEM. In the same publication, we identified the terms STEAM, STEM and STEM education, whose frequency is 1, 4 and 5, respectively.
Identification of Terms to Be Included in the New Taxonomy

After reviewing the synonymy of the 8560 key terms in the indexed journals, we established that 245 of them were relevant to the discipline. Before generating the final list of relevant terms to be included, we performed a new review of the synonymy among them. For example, we unified, in the term *Learning types* (with an absolute frequency of 15) the following terms:

- Adaptive learning
- Experiential learning
- Exploitative learning
- Explorative learning
- Exploratory learning
- Inquiry-based learning
- Interactive learning
- Learning strategy
- Project-based learning
- Self-Regulated learning
- Self-Regulated learning ability
- Situated learning

After the debugging, we established the final list of relevant terms that should be included in the new taxonomy. We took into account the absolute frequency of the key terms in the list. We decided that the 30 terms with the highest frequency would be included. These terms have a relative frequency of more than 0.5%, in relation to the sum of the frequencies of the terms we initially identified as relevant. We present in Table 3 the list of the first 10 terms that were included in the new taxonomy, by their frequency of use in the indexed journals.

<table>
<thead>
<tr>
<th>Term</th>
<th>Frequency</th>
<th>Relative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teacher knowledge</td>
<td>132</td>
<td>21.78%</td>
</tr>
<tr>
<td>Teaching practice</td>
<td>61</td>
<td>10.07%</td>
</tr>
<tr>
<td>Communication</td>
<td>33</td>
<td>5.45%</td>
</tr>
<tr>
<td>Achievement</td>
<td>26</td>
<td>4.29%</td>
</tr>
<tr>
<td>International studies</td>
<td>21</td>
<td>3.47%</td>
</tr>
<tr>
<td>Conceptual change</td>
<td>19</td>
<td>3.14%</td>
</tr>
<tr>
<td>Perceptions</td>
<td>16</td>
<td>2.64%</td>
</tr>
<tr>
<td>Learning types</td>
<td>15</td>
<td>2.48%</td>
</tr>
<tr>
<td>Mathematics literacy</td>
<td>13</td>
<td>2.15%</td>
</tr>
<tr>
<td>Decision making</td>
<td>13</td>
<td>2.15%</td>
</tr>
</tbody>
</table>
Selection of Terms that Make Up the New Taxonomy

To elaborate the new taxonomy, we took as a basis the list of terms that make up the existing taxonomy (Gómez & Cañadas, 2013). Additionally, we included the terms that we identified as relevant due to their frequency of use in the journals indexed in Scopus and Web of Science. The inclusion of these terms implies the elimination of the same amount of terms from the existing taxonomy, since we were interested in conserving the quantity of terms between 350 and 400. We took into account the frequency of use of the key terms of this taxonomy in the Funes repository.

As previously stated, we decided to include the 30 most frequently used terms in the indexed journals. Their relative frequency, in relation to the sum of the frequencies of the terms in the list of relevant terms, is over 0.5%. This led us to the neglect of terms from the existing taxonomy that are little used in practice. We determined the measure of use of the terms of that taxonomy according to the frequencies of the key terms in the repository.

We took the list of key terms from the Funes repository on March 6th, 2020. At that date, the repository contained approximately 12000 records. We organized the key terms in ascending order according to their frequency. We selected 30 terms for their low use (their relative frequency in relation to the sum of the frequencies of all key terms in the repository is less than 0.03%) and decided to omit them for the new taxonomy.

There is no relationship between these relative frequencies and the relative frequency we identified as a limit in the list of key terms in the indexed journals. We took into account an equivalent amount of terms that are included in the new taxonomy and terms that are omitted from the existing taxonomy. We present in Table 4 the list of the first 10 terms omitted from the new taxonomy.

<table>
<thead>
<tr>
<th>Term</th>
<th>Frequency</th>
<th>Relative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Other departments</td>
<td>1</td>
<td>0.002%</td>
</tr>
<tr>
<td>Staff</td>
<td>1</td>
<td>0.002%</td>
</tr>
<tr>
<td>Other educational centers</td>
<td>1</td>
<td>0.002%</td>
</tr>
<tr>
<td>Non-regulated</td>
<td>1</td>
<td>0.001%</td>
</tr>
<tr>
<td>Financial</td>
<td>2</td>
<td>0.003%</td>
</tr>
<tr>
<td>Longitudinal studies</td>
<td>3</td>
<td>0.004%</td>
</tr>
<tr>
<td>Infrastructure</td>
<td>4</td>
<td>0.005%</td>
</tr>
<tr>
<td>Customized</td>
<td>4</td>
<td>0.005%</td>
</tr>
<tr>
<td>Transversal study</td>
<td>5</td>
<td>0.006%</td>
</tr>
</tbody>
</table>

In reviewing the frequencies of all key terms in the Funes repository, we found terms that refer to “other” have high frequencies. Therefore, we checked in detail which terms that do not appear in the existing taxonomy are
relevant to the community and should be included in the new taxonomy. In Table 5, we present each “other” term we had to go into in depth, its absolute frequency, its relative frequency, and the new terms that emerged from it.

<table>
<thead>
<tr>
<th>Initial term</th>
<th>Frequency</th>
<th>Relative frequency</th>
<th>New key terms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Other (learning theory)</td>
<td>80</td>
<td>0.102%</td>
<td>APOE, Socio-epistemological approach, Ontosemiotic approach</td>
</tr>
<tr>
<td>Other (types of study)</td>
<td>97</td>
<td>0.125%</td>
<td>Pre-experimental study, Exploratory study, Documental study, Descriptive study</td>
</tr>
<tr>
<td>Other (methodologies)</td>
<td>492</td>
<td>0.782%</td>
<td>Qualitative, Quantitative, Mixed, Content analysis</td>
</tr>
</tbody>
</table>

In our proposal, we decided to retain the groups of key terms called focus and educational level in the existing taxonomy. We organized the discipline-specific terms into three groups: foundations of Mathematics Education, research in Mathematics Education and pedagogical notions. We included the key terms of the mathematical content in one single category, without differentiating terms for school mathematics and terms for higher mathematics.

**Validation of the New Taxonomy**

The relevance of the new taxonomy was validated by researchers and innovators in Mathematics Education who participated in a process of triangulation of information. The revision of the taxonomy in its two versions (English and Spanish) was carried out from the list of key terms, organized hierarchically. Suggestions and comments for the adjustment of the proposal were recorded there. On one side of the list, we established a column of suggestion, in which, for each term, a list with these options was displayed:

(a) It must have another tag,
(b) It must be removed, and
(c) It must be in another section.

Each reviewer was able to select one of these options if required. In another column, we invited them to record in the column called Comments the reason for the suggestion (see Figure 4).

In addition, we provided a space for the registration of suggestions for terms that should be included in the taxonomy. However, the recommendation was that, if so, it should be indicated which term from the list we
provided initially should be removed. We were interested in retaining the length of the taxonomy, given our purpose of its practical use for coding, organizing and searching documents.

![Figure 4. Outline for the Suggestion Log for Each Term](image)

Initially, we consulted experts from local research teams on the structure of the taxonomy and the labelling of key terms. Some of the comments they provided related to the lack of clarity of several terms. For example, with regard to the assessment, one expert stated that “the difference between types and purposes is not found. This term could be clarified with some options”. This led us to omit the Purposes label and use Assessment modalities, while referring to self-assessment, co-assessment and peer assessment, among other possible modalities. Another recommendation was to modify the International standardized label and restrict it to Standardized “to include national tests”. The comments of these researchers also led us to review the organization of the mathematical content. As one of the results, we organized the terms of Statistics into two sections: descriptive and inferential.

Based on the comments received for version 1, we prepared the second version of the taxonomy. This version was sent to internationally recognized experts in Mathematics Education. We received suggestions to include, join or delete terms. However, these recommendations were only considered when verifying the relevance of the terms by their frequency of use. An example of this is related to the types of research: “perhaps the terms ‘design or action research’ should be joined together as they are not disjointed”. In this respect, we verified the frequency of these terms in the open access repository and decided to keep the two key terms Design research and Action research. In relation to the labels, we followed suggestions that would allow the terms not to be restricted to a particular theoretical approach. In this sense, we changed the term Representation systems to Representations, as it could be “restrictive and in the end if someone searches for this subject, they usually search by ‘representations’. Although there is a theoretical justification for this term to appear, it is not practical”. We also adjusted the labels that refer to teacher associations, tasks, problems and special educational needs.

Version 3 of the taxonomy was sent to the editors of the 33 Mathematics Education journals that have JCR and/or SJR impact factors—a list of these publications is available at https://bit.ly/3hBZTJZ. Additionally, we contacted experts attached to organizations that lead international events in the discipline. The purpose of this
last phase of taxonomy review was to validate its relevance and effectiveness in assigning key terms in articles or contributions to event reports. The comments of this group of experts led us to make adjustments to the structure and some labels of the terms. Regarding the first case, we placed Affectivity as a first level term in the category of Pedagogical notions, since it appeared as a second level term, emerging from Cognition, in the previous versions. We placed the term Analytical geometry at the same level as Trigonometry and Topology, and we omitted the inclusion of a term called recordings, since it is immersed as an instrument in the Interviews and Classroom observations that we propose in the Sources of information label. On the other hand, we included full expressions for terms such as Justification processes (which includes both argumentation and demonstration) and Mathematical analysis (not only analysis), as far as the mathematical content is concerned.

In accordance with the procedures set out above, we present below the structure of the new taxonomy of key terms in Mathematics Education.

**Structure of the New Taxonomy**

The new taxonomy is organized into six categories of key terms: (a) purpose, (b) educational level, (c) foundations of Mathematics Education, (d) research in Mathematics Education, (e) pedagogical notions and (f) mathematical content. The categories’ purpose and educational level only include first level terms.

**Key Terms Associated with Purpose**

The key terms associated with *purpose* are task, essay, innovation and research. A task is a stimulus on a specific topic that seeks to promote learning in the classroom. An essay is the presentation of an opinion or position, which does not require systematic processes of justification. An innovation is a curriculum design based on disciplinary knowledge. A research is a work that makes a contribution to the knowledge that emerges from a systematic process of inquiry.

**Key Terms Associated with Educational Level**

The key terms corresponding to the *level of education* are as follows.

- Kindergarten, early childhood education, pre-school education (0 to 6 years)
- Primary education, elementary school (6 to 12 years)
- Secondary education, middle school, basic secondary (12 to 16 years)
- High school, upper secondary (16 to 18 years old)
- Technical education, vocational education, professional training
- Continuing education
- Higher education, undergraduate education
- Postgraduate education
- Adult education
• All educational levels
• No educational level
• Other education level

Key Terms Associated with Foundations of Mathematics Education

The category of key terms related to the foundations of Mathematics Education are organized in five first level terms, from which second level terms are derived, as follows.

• History of Mathematics Education
• From academic disciplines
  o Epistemology
  o Pedagogy
  o Psychology
  o Semiotics
  o Sociology
• Conceptual frameworks
  o French didactics
  o Ethnomathematics
  o Critical Mathematics Education
  o Sociological theories
• Models
  o Didactical analysis
  o Ontosemiotic approach
  o Socio-epistemological approach
• Purposes of Mathematics Education
  o Cultural
  o Formative
  o Socio-political

Key Terms Associated with Research of Mathematics Education

The category Research in Mathematics Education contains 31 key terms, of which four are first level, 18 are second level and 9 are third level. The hierarchy of terms in this category is as follows.

• Paradigms
  o Critical
  o Empirical-analytical
  o Interpretative
• Type of research
  o Theoretical
  o Empirical
  o Qualitative
  o Quantitative
  o Mixed

• Methods
  o Content analysis
  o Discourse analysis
  o Meta-analysis
  o Statistical methods
  o Types of studies
    ▪ Comparative
    ▪ Quasi-experimental
    ▪ Pre-experimental
    ▪ Case studies
    ▪ Exploratory
    ▪ Descriptive
    ▪ Documental
    ▪ Design research
    ▪ Action research

• Sources of information
  o Surveys
  o Interviews
  o Tests
  o Questionnaires
  o Classroom observations

**Key Terms Associated with Pedagogical Notions**

The key terms in the category of *pedagogical notions* are organized in 10 first level terms: (a) educational system, (b) educational center, (c) teacher, (d) content, (e) learning, (f) cognition, (g) teaching, (h) assessment, (i) inclusion and (j) affectivity. Terms referring to learning, cognition, assessment and affectivity are not exclusive to the student; they can be associated with the teacher.

• Educational system
  o Educational policy
  o Educational laws
  o Curricular documents
  o Management and quality
• Educational center
  o Management and organization
  o Center resources
  o Educational project of the center

• Teacher
  o Associations
  o Teacher’s knowledge
  o Teacher development
    ▪ Status
    ▪ Identity
  o Peer collaboration
  o Teacher practice
  o Training
    ▪ Initial
    ▪ Postgraduate
    ▪ Continuing education

• Content
  o Approaches
    ▪ Conceptual-theoretical
    ▪ Functional
  o Concepts and procedures
  o Representations
    ▪ Graphical
    ▪ Numerical
    ▪ Symbolic
    ▪ Verbal
  o Phenomenology
    ▪ Uses or meanings
    ▪ Contexts or situations
  o Historical evolution of concepts

• Learning
  o Learning theories
    ▪ Constructivism
    ▪ Socio-cultural location
    ▪ Social theory of learning
    ▪ Objectification theory
    ▪ APOS
  o Learning types
Learning goals
- Mathematical literacy
- Competences
- Capacities

Learning limitations
- Misconceptions
- Difficulties
- Errors

Learning styles

Achievement

Cognition
- Conceptual change
- Knowledge
- Metacognition

Cognitive processes
- Estimation
- Mental calculation
- Perception
- Visualization
- Objectivation
- Understanding
- Abstraction
- Generalization
- Reasoning
  - Deductive
  - Inductive
- Justification processes
- Creativity

Problem solving
- Formulate
- Modeling
- Interpret
- Communication

Types of mathematical thoughts

Teaching
- Curriculum
  - Notion
  - Design
  - Development
  - Evaluation
- Types of methodology
- Tasks
  - Routine exercises
  - Problems
    - Problem statement
    - Solution strategies
    - Types of problems
- Didactic resources
  - Textbooks
  - Manipulative materials
  - Audio-visual media
  - Software
  - Electronic devices
    - Calculators
    - Computers
    - Mobile devices
- Classroom management
  - Decision making
  - Discourse
  - Socio-cultural norms
  - Interactions
- Reflection on teaching
- Teaching modality
  - Face-to-face
  - Virtual
  - Mixed
  - Distance education
- Assessment
  - Types of assessment
    - Formative
    - Summative
    - Diagnostic
  - Instruments
  - Assessment approaches
  - Standardized
  - Feedback
- Inclusion
  - Socio-cultural diversity
    - Socioeconomic aspects
    - Culture / Religion
Key Terms Associated with Mathematical Content

Finally, we organized the key terms of the mathematical content according to a phenomenological approach. We used the content categories proposed in the conceptual framework of the PISA 2015 study (OECD, 2016): quantity, change and relations, space and shape, and uncertainty and data. We included the term STEM and generated a category containing content that addresses the previous categories in a transversal way. We present, in what follows, the hierarchical organization of the terms.

- Quantity
  - Numbers
    - Pre-numerical concepts
    - Numbering systems
    - Numerical sets
      - Natural numbers
      - Integer numbers
      - Rational numbers
        - Fractions
        - Decimals
      - Irrational numbers
      - Real numbers
      - Complex numbers
    - Arithmetic operations
      - Addition
      - Subtraction
      - Multiplication
      - Division
    - Numerical relations
      - Order
• Divisibility
• Proportionality
  o Ratios
  o Proportions
  ▪ Number theory
  o Measure
    ▪ Magnitudes
    ▪ Units of measurement
    ▪ Measurement estimation
    ▪ Measurement calculation
• Change and relationships
  o Algebra
    ▪ Numerical patterns
    ▪ Polynomials
    ▪ Relations
    ▪ Functions
    ▪ Types
      o Exponential
      o Polynomial
      o Logarithmic
      o Trigonometric
  • Operations
    ▪ Equations and inequalities
    ▪ Systems of equations
    ▪ Abstract algebra
  o Calculation
    ▪ Sequences and series
    ▪ Limits
    ▪ Derivation
    ▪ Integration
  o Differential equations
  o Mathematical analysis
• Space and shape
  o Dimensions
    ▪ One-dimensional
    ▪ Two-dimensional
    ▪ Three-dimensional
  o Geometry
    ▪ Geometric constructions
    ▪ Geometric shapes
- Geometrical transformations
- Geometric relations
- Theorems
  - Analytical geometry
  - Trigonometry
  - Linear algebra
  - Topology
- Uncertainty and data
  - Statistics
    - Descriptive
      - Data organization and representation
      - Measures of central tendency
      - Measures of dispersion
    - Inferential
      - Random variable
      - Correlation and regression
      - Hypothesis testing
  - Probability
    - Combinatorics
    - Probability calculation
    - Conditional probability
    - Probability distributions
- Transversal
  - Mathematical logic
  - Set theory
  - Graph theory
- STEM

Conclusions

We present in this document the systematic process of production and validation of a new taxonomy of specific key terms in Mathematics Education. This proposal addresses the current state of knowledge in the discipline, as it is consolidated from the frequency of use of key terms in specialized research journals and in an open access digital document repository. We took as a basis an existing taxonomy, which emerged from a concrete theoretical approach and with a specific purpose of use (Gómez & Cañadas, 2013). We eliminated from that taxonomy the less used terms in the document repository and included the terms that are relevant because of their frequency in the journals indexed in Scopus and Web of Science. The revision of encyclopedias specific to Mathematics Education (Grinstein & Lipsey, 2001; Lerman, 2014) supported the classification of terms as relevant and guided us in the hierarchical organization of the taxonomy.
We carried out a three-stage validation process in which we invited experts in the discipline to evaluate the structure, relevance and usefulness of the taxonomy. These comments allowed us to make the proposal more concrete. It should be clarified that, although the experts’ suggestions were considered, the terms that finally made up the taxonomy satisfy the criterion of being relevant because of their use in the databases and the open access document repository.

Compared to the classification of terms used in MathEduc (FIZ Karlsruhe, 2019), our proposal focuses on Mathematics Education and provides a hierarchy of key terms, organized by main categories, which facilitates the coding and search of documents. In relation to the use of Bloom's taxonomy to establish descriptors associated with learning mathematics (Long & Dunne, 2014; Radmehr & Drake, 2019), the new taxonomy starts from the current knowledge of the discipline to organize key terms associated with pedagogical notions such as teacher, learning, cognition, teaching, assessment and affectivity. It is a fact that Mathematics Education, at present, is not limited to the study of the cognitive dimension of the student, but includes in its research and innovation agenda other aspects of mathematics teaching, teacher training and development, educational policy and the affective dimension (Lerman, 2020).

We start from the knowledge included in specialized journals in Mathematics Education to identify the terms that are relevant to the discipline and add them to the taxonomy proposed by Gómez and Cañadas (2013). For example, in the articles in the Web of Science database, aspects such as theoretical frameworks of the discipline, and teaching and cognitive processes such as generalization are evidenced as trends in Mathematics Education (Gökçe & Guner, 2021). These terms are included in our proposal. In the same way, we identified the terms of this taxonomy that are less used as document descriptors, which led us to refine it. The quantitative analysis carried out to make decisions about the key terms that are relevant in the discipline is supported by the fact that numerical trends highlight the systemic and systematic practices of the international community of mathematics educators (Young & Young, 2022).

The production of the new taxonomy is aligned with reflections on the usefulness of this type of controlled vocabulary, because of its effectiveness in organizing, managing and searching for information from tags that characterize knowledge in a field (Sujatha et al., 2011). Our proposal starts from conceptual references typical of Mathematics Education to establish the categories that organize the key terms. This approach has been used in other academic disciplines to develop their own taxonomies from existing references and resources (Aadland & Aaboen, 2020; Klassen & Donald, 2020). Bibliographic review in databases (strategy used in our proposal) has served as a basis in other studies to identify and structure disciplinary knowledge in specific categories (Fellnhofer, 2019; Pertegal-Vega et al., 2019).

We recognize some limitations of our proposal. First, we restricted the review of key terms to journals in the discipline that, as of 2017, were indexed in both databases. This meant that we left aside publications that only satisfied the condition of being indexed in Scopus or that, although not specific to Mathematics Education, published documents from the discipline (for example, Revista Enseñanza de las Ciencias). To solve this situation, we considered it important to invite the editors of all the journals currently (2020) indexed in Web of
Science, Scopus and Emerging Sources Citation Index to validate the taxonomy according to their experience and knowledge in the discipline. In addition, we are aware that the taxonomy may be limited to characterize the knowledge produced in Mathematics Education that is manifested in the documentation that is disseminated through various dissemination schemes. To go into more detail would imply a substantially greater amount of key terms. We wanted to keep the size of the previous taxonomy and to include in it the terms that may have greater frequency in the current documents of our discipline and those that will be published in the short and medium term.

We believe that the taxonomy presented in this document manages to synthesize the current focuses of work in Mathematics Education at the international level. Its structure and extension facilitate the codification, organization and search of documents in different dissemination schemes, such as journals, event pages or specific databases. However, this proposal is not rigid and is susceptible to revision and adjustment as changes in the thematic trends of the academic community are recognized.

In fact, the procedures we develop to produce and validate the taxonomy can be used to generate new proposals. We emphasize that the method we propose is also useful in other disciplines. In general, in the characterization of knowledge it is relevant to identify the advances that arise from research (through, for example, the publications with the greatest impact) but it is also important to study the documentation that is disseminated in an open manner and that is not restricted to research results (Castro & Gómez, 2021). In the case of Mathematics Education, the identification of trends manifested in the documentation produced by the international community also provides opportunities for the development of future researches (Gökçe & Guner, 2021).

We make the taxonomy available to the Mathematics Education community under license the Creative Commons Attribution–NonCommercial–NoDerivs License. On the https://bit.ly/3f6ffVA website, the taxonomy can be downloaded in English and Spanish, and in different formats.

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References


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